The constitutive equation for electric displacement is

$$D_3 = e_{31}\frac{\partial u}{\partial x} + \epsilon_3 E(x,t) \qquad (1)$$

Up till now we were essentially working off the stress constitutive equation.

The displacement current through the PZT may be calculated using Eq (1).

$$i(t) = \int_{A_e} \dot{D}_3 \, dA_e \qquad (2)$$

where A_e is the area of the electrodes, assumed to be symettrical on the top & bottom, thus,

$$dA_e = b\,dx \qquad (2.1)$$

so

$$i(t) = b\int_{x_1}^{x_2} \dot{D}\,dx \qquad (3)$$

Hence, from (1) & (3)

$$i(t) = b\left[e_{31}\left(\dot{u}(x_2,t) - \dot{u}(x_1,t)\right) + \epsilon_3\int_{x_1}^{x_2} \dot{E}(x,t)\,dx\right] \qquad (4)$$

We have previously defined

$$E(x,t) = -\frac{V_0}{h}\left(H(x-x_1) - H(x-x_2)\right) \mathrm{Im}\{e^{j\omega t}\} \qquad (5)$$

~~So (4) becomes~~ Noting

$$\epsilon_3 \int_{x_1}^{x_2} \dot{E}(x,t) = -\frac{\epsilon_3 V_0}{h} \omega j \,\mathrm{Im}\{e^{j\omega t}\} \int_{x_1}^{x_2} H(x-x_1) - H(x-x_2)\, dx \qquad (6a)$$

Noting that

$$\int_{x_1}^{x_2} (H(x-x_1) - H(x-x_2))\, dx = \int_{x_1}^{x_1} 1\, dx = x_2 - x_1 \qquad (6b)$$

(6) becomes

$$\epsilon_3 \int_{x_1}^{x_2} \dot{E}(x,t) = \frac{-\epsilon_3 V_0 (x_2 - x_1)}{h} \omega j \,\mathrm{Im}\{e^{j\omega t}\} \qquad (7)$$

Also

$$A_e = b(x_2 - x_1)$$

we have

$$b\, \epsilon_3 \int_{x_1}^{x_2} \dot{E}(x,t)\, dx = \frac{\epsilon_3 A_e}{h} V_0 \omega j \,\mathrm{Im}\{e^{j\omega t}\} = \ldots$$

$$\ldots = -c_0 V_0 \omega j \,\mathrm{Im}\{e^{j\omega t}\} \qquad (8)$$

where

$$c_0 = \frac{\epsilon_3 A_e}{h} \qquad (9)$$

represents the so-called blocking capacitance of the PZT element

Remember from DNo. 3 eq(14), the steady state solution may be written

$$u(x,t) = \sum_{c=1}^{\infty} |X_c| \, \mathrm{Im}\{e^{j(\omega t - \phi_c)}\} U_c(x) \quad (10)$$

so

$$be_{31}(u(x_2,t) - u(x_1,t)) = be_{31}\sum_{c=1}^{\infty} |X_c| \, \mathrm{Im}\{e^{j(\omega t - \phi_c)}\} \cdot \ldots \quad (11)$$

$$\ldots \times (U_c(x_2) - U_c(x_1))\,\omega$$

Noting (8) & (11), (4) becomes

$$i(t) = \left[be_{31}\sum_{c=1}^{\infty} |X_c| \, \mathrm{Im}\{e^{j(\omega t - \phi_c)}\}(U_c(x_2) - U_c(x_1)) - C_0 V_0 \, \mathrm{Im}\{e^{j\omega t}\}\right]\omega j \quad (12)$$

$$i(t) = \left[be_{31}\sum_{c=1}^{\infty} |X_c| \, \mathrm{Im}\{e^{-j\phi_c}\}(U_c(x_2) - U_c(x_1)) - C_0 V_0\right] \mathrm{Im}\{e^{j\omega t}\}\,\omega j \quad (13)$$

$$i(t) = \underbrace{\left[be_{31}\sum_{c=1}^{\infty} |X_c| \sin(-\phi_c)(U_c(x_2) - U_c(x_1)) - C_0 V_0\right]}_{Z} \cos(\omega_c r_c t)\,\omega_c r_c \quad (14)$$

Thus the admittance can be written as

$$\tilde{Y}(r_n) = \frac{i(t)}{\phi(t)} = \frac{\omega_n r_n Z \cos(\omega_n r_n t)}{V_0 \sin(\omega_n r_n t)} \quad (15)$$

$$\tilde{Y}(r_n) = \frac{\omega_n r_n Z}{V_0} \cot(\omega_n r_n t) \quad (16)$$