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Electric Field and Plasma Flow: What Drives What?

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Abstract. The MHD approximation connects the plasma bulk flow velocity and the electric field, but it does not say whether one of them can be considered as causing or producing the other, and if so, which one. This question is often viewed as one having no unambiguous answer and possibly no physical meaning. However, a definite answer can be obtained by solving the basic equations with appropriate initial values, with the result that, for the commonly considered case where the Alfvén speed is small compared to the speed of light, (1) a given plasma bulk flow produces an electric field, (2) a given electric field does not produce a plasma bulk flow. The general result can also be derived as a simple consequence of conserving the total (plasma plus electromagnetic field) linear momentum.

Introduction

The MHD approximation

$$c\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0, \tag{1}$$

widely applied in many situations in plasma physics, implies a one-to-one relation between the electric field $\mathbf{E}$ and the bulk flow velocity of the plasma $\mathbf{V}$: given the value of either one, the other must have the corresponding value given by (1). By itself, however, the relation does not say whether either one may be regarded as causing or producing the other in a physical sense — an infrequently raised question that does not seem to have a generally accepted answer. Often, language that presupposes a specific answer is used in papers casually and uncritically (e.g. "...the electric fields that give rise to bursty flows..." [*Lyons et al.*, 1999]). When the question is explicitly asked, a common reply is that there is really no unique, physically meaningful answer: it's largely semantics, it all depends on what approach to describing plasmas one has adopted and what one's views are in the ongoing controversy [*Parker*, 1996, 1997, 2000; *Heikkila*, 1997; *Lui*, 2000] on whether the magnetic field and the plasma flow or the electric current and the electric field are to be treated as the primary variables. In this Letter I show that, on the contrary, the question is not one of semantics nor of choice of paradigm but can be given a definite and unambiguous answer from the equations of physics.

Basic Approach

What makes it possible to discuss such questions on the basis of purely physical rather than philosophical or semantic arguments is a remarkable property of classical (non-quantum) physics: all of its governing equations, except for three, can be written in the evolutionary form

$$\partial Q_k/\partial t = F_k(Q_1, Q_2, Q_3, ...) \tag{2}$$

where the Q's are all the quantities describing the system and the F's are functions of the Q's and their spatial derivatives at a given time. The three exceptions are the divergence equations of the electromagnetic and gravitational fields,

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 4\pi\rho_c \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \cdot \mathbf{g} &= -4\pi G\rho \ . \end{aligned} \tag{3}$$

The significance of this formulation (so familiar as to be hardly ever mentioned explicitly in textbooks; it was impressed on me by my thesis supervisor at M.I.T., Prof. Stanislaw Olbert) is that all the time derivatives are determined, solely and completely, by values at the present time. Furthermore, any initial conditions whatsoever, provided only that they satisfy the divergence equations (3), can be imagined at an instant of time (but only at that instant), and the equations will then determine what happens at all other times. The question in the title of this Letter can thus be answered by means of two thought experiments: at the initial instant assume, in one case an electric field but no plasma bulk flow, in the other a flow but no electric field, and use the equations to determine the subsequent evolution of field and flow in both cases (needless to say, the exact equations must be used and not the MHD approximation).

Mathematical Development

Consider a system that initially is homogeneous with uniform magnetic field $\mathbf{B}$ and plasma with mass density ρ and electron concentration n. Spatial homogeneity is assumed not just for simplicity but also to ensure that the looked-for role of the electric field in producing the flow is not swamped by the potentially much larger effects of stresses from gradients. To exclude any influence from boundary conditions, the initially homogeneous system is taken to extend out to a distance R_2 from the origin, but we will be interested only in the region out to $R_1 < R_2$ and in the time interval $0 \leq t < \tau$, where

$$\tau = (R_2 - R_1)/c \tag{4}$$

so that no physical effects from the boundary have had time to reach the region of interest. Since light travels a distance of $2\pi\lambda_e$ in one plasma oscillation period, where λ_e is the electron inertial length (collisionless skin depth), and since R_1 and $R_2 - R_1$ must be very large compared to λ_e for MHD to be applicable at all, the time interval τ is very long compared to the period of plasma oscillations, which turns out to be amply adequate for our purpose.

Assume as initial conditions at time $t = 0$ the current density $\mathbf{J} = 0$, $\mathbf{E} = \mathbf{E}_0$, and $\mathbf{V} = \mathbf{V}_0$, where $\mathbf{E}_0$ and $\mathbf{V}_0$ are perpendicular to $\mathbf{B}$ but otherwise arbitrary and do *not* satisfy equation (1). The equations governing the evolution of the system are (in Gaussian units and standard notation)

$$\partial\mathbf{B}/\partial t = -\nabla \times c\mathbf{E} \tag{5}$$

$$\partial\mathbf{E}/\partial t = -4\pi\mathbf{J} + \nabla \times c\mathbf{B} \tag{6}$$

$$\partial\mathbf{J}/\partial t = (ne^2/m)(\mathbf{E} + \mathbf{V} \times \mathbf{B}/c - \mathbf{J} \times \mathbf{B}/nec) + \cdots \tag{7}$$

$$\partial\mathbf{V}/\partial t = \mathbf{J} \times \mathbf{B}/\rho c + \cdots \tag{8}$$

(plus continuity equations which turn out to be unnecessary in the present case). Equations (5) and (6) are Maxwell's equations and hence exact. Equation (7) is the generalized Ohm's law (see e.g. *Rossi and Olbert* [1970] and (8) the momentum equation for a two-component plasma of electrons (mass m) and ions (mass M, $\rho = nM$); both are exact except for neglecting terms of order m/M and writing $+\cdots$ for all the spatial-derivative terms (pressure gradients, etc.).

Because of the assumed initial spatial homogeneity, at $t = 0$ all the spatial derivatives vanish, including $\nabla \times \mathbf{E}$ and $\nabla \times \mathbf{B}$. The time derivatives are then the same at all points within the region under consideration, and spatial homogeneity is therefore preserved at later times as well. This implies that $\nabla\times\mathbf{E}$ and $\nabla\times\mathbf{B}$ remain zero, hence by (5) $\mathbf{B}$ remains constant. (For the same reason of spatial homogeneity, densities do not change and the continuity equations are not needed.) To solve for $\mathbf{E}$, $\mathbf{J}$, and $\mathbf{V}$, which are now functions of time only, differentiate (7), use (6) and (8) to eliminate the time derivatives of $\mathbf{E}$ and $\mathbf{V}$, and obtain an equation for $\mathbf{J}$ alone:

$$d^2\mathbf{J}/dt^2 + {\omega_p}^2[\mathbf{J} + ({V_A}^2/c^2)\mathbf{J}_\perp] + d\mathbf{J}/dt \times e\mathbf{B}/mc = 0 \tag{9}$$

where ω_p is the (electron) plasma frequency and V_A the Alfvén speed; note that

$${\omega_p}^2{V_A}^2/c^2 = \Omega_i\Omega_e \tag{10}$$

with Ω_i, Ω_e the ion and electron gyrofrequencies. Equation (9) is to be solved subject to the initial values at $t = 0$

$$\begin{aligned} \mathbf{J} &= 0 \\ d\mathbf{J}/dt &= ({\omega_p}^2/4\pi)(\mathbf{E}_0 + \mathbf{V}_0 \times \mathbf{B}/c) \end{aligned} \tag{11}$$

The component of (9) along $\mathbf{B}$ is

$$d^2J_\parallel/dt^2 + {\omega_p}^2 J_\parallel = 0 \tag{12}$$

whose solution subject to (11) is $J_\parallel = 0$. The solution for the perpendicular components can be obtained by standard Fourier analysis techniques, yielding with the initial conditions (11)

$$\mathbf{J} = \mathbf{J}_1(\sin\omega_+ t + \sin\omega_- t) + \mathbf{J}_1 \times \mathbf{b}(\cos\omega_+ t - \cos\omega_- t) \tag{13}$$

where

$$\mathbf{J}_1 \equiv [{\omega_p}^2/4\pi(\omega_+ + \omega_-)](\mathbf{E}_0 + \mathbf{V}_0 \times \mathbf{B}/c)\ , \tag{14}$$

$\mathbf{b}$ is the unit vector along $\mathbf{B}$, and the characteristic frequencies ω_+ and ω_- are given by

$$\omega_\pm = ({\omega_p}^2 + \Omega_i\Omega_e + \Omega_e^2/4)^{1/2} \pm \Omega_e/2\ . \tag{15}$$

Not surprisingly, these are the frequencies given by the dispersion relation for waves in cold plasmas in the limit of infinite wavelength (see e.g. *Stix* [1962]).

Given the solution for $\mathbf{J}$, solutions for $\mathbf{E}$ and $\mathbf{V}$ are obtained by integrating (6) and (8), respectively, with the results

$$\mathbf{E} = \mathbf{E}_m - 4\pi \int dt\mathbf{J} \tag{16}$$

$$\mathbf{V} = \mathbf{V}_m - (\mathbf{B}/\rho c) \times \int dt\mathbf{J} \tag{17}$$

where

$$\begin{aligned} \int dt\mathbf{J} = &-\mathbf{J}_1[(\cos\omega_+ t)/\omega_+ + (\cos\omega_- t)/\omega_-] \\ &+ \mathbf{J}_1 \times \mathbf{b}[(\sin\omega_+ t)/\omega_+ - (\sin\omega_- t)/\omega_-] \end{aligned} \tag{18}$$

is the oscillating (zero mean) function obtained by integrating (13), and $\mathbf{E}_m$, $\mathbf{V}_m$ are the steady mean values resulting from choosing the constants of integration to give the assumed initial values:

$$\mathbf{E}_m = [({V_A}^2/c^2)\mathbf{E}_0 - \mathbf{V}_0 \times \mathbf{B}/c]/(1 + {V_A}^2/c^2) \tag{19}$$

$$\mathbf{V}_m = [\mathbf{V}_0 + ({V_A}^2/c^2)c\mathbf{E}_0 \times \mathbf{B}/B^2]/(1 + {V_A}^2/c^2)\ . \tag{20}$$

In most applications within space plasma physics, ${V_A}^2/c^2 \ll 1$.

Results

We have now obtained the solutions describing the behavior of a locally homogeneous plasma after arbitrary initial values of the perpendicular electric field and plasma bulk flow, not satisfying the MHD approximation (1), have been imposed. $\mathbf{J}$, $\mathbf{E}$, and $\mathbf{V}$ all undergo oscillations at frequencies just above the plasma frequency; in addition, $\mathbf{E}$ and $\mathbf{V}$ assume mean values related to the initial values by (19) and (20). It is easily verified that the mean values satisfy the MHD approximation (1), as expected.

Consider, as special cases, the two extremes:

(1) Initially only an electric field $\mathbf{E}_0$ is imposed and no flow: the resulting mean values are

$$\mathbf{E}_m = ({V_A}^2/c^2)\mathbf{E}_0/(1 + {V_A}^2/c^2) \tag{21}$$

$$\mathbf{V}_m = ({V_A}^2/c^2)(c\mathbf{E}_0 \times \mathbf{B}/B^2)/(1 + {V_A}^2/c^2) \tag{22}$$

The mean electric field has been reduced to a small fraction ${V_A}^2/c^2$ of the initial value (the instantaneous field oscillates between the initial value and, very nearly, its negative), and only a correspondingly small plasma flow has been created.

(2) Initially only a plasma bulk flow $\mathbf{V}_0$ is imposed and no electric field: the resulting mean values are

$$\mathbf{E}_m = -(\mathbf{V}_0 \times \mathbf{B}/c)/(1 + {V_A}^2/c^2) \tag{23}$$

$$\mathbf{V}_m = \mathbf{V}_0/(1 + {V_A}^2/c^2) \tag{24}$$

The mean flow has remained at nearly its initial value, and a mean electric field equal to $-\mathbf{V} \times \mathbf{B}/c$ has been created (the instantaneous field oscillates between its initial value of zero and twice its mean value).

Alternative Derivation From Conservation of Momentum

A much simpler derivation of the mean values, which, moreover, makes their physical meaning more apparent, becomes possible if we assume that, whatever the initial values of $\mathbf{E}$ and $\mathbf{V}$, their final mean values satisfy the MHD approximation (1). The linear momentum density is

$$\mathbf{G} = \rho\mathbf{V} + \mathbf{E} \times \mathbf{B}/4\pi c \tag{25}$$

where the first term represents the momentum of plasma bulk flow and the second that of the electromagnetic field. When $\mathbf{E}$ and $\mathbf{V}$ are related by the MHD approximation (1), the momentum density can be rewritten in two equivalent forms

$$\begin{aligned} \mathbf{G} &= \rho\mathbf{V}(1 + {V_A}^2/c^2) \\ &= (\mathbf{E} \times \mathbf{B}/4\pi c)(1 + c^2/{V_A}^2) \; . \end{aligned} \tag{26}$$

In the present case, spatial homogeneity implies that linear momentum is conserved locally, and therefore the initial value (given by (25) with the initial $\mathbf{E}_0$ and $\mathbf{V}_0$) must equal the final value (given by either form of (26) with the mean $\mathbf{E}_m$ and $\mathbf{V}_m$), which yields expressions for $\mathbf{E}_m$ and $\mathbf{V}_m$ identical with (19) and (20).

Whatever linear momentum has been imposed on the plasma initially must, in the final MHD regime, be shared between electromagnetic field and plasma bulk flow in the ratio given by (26) as

$$(\mathbf{E} \times \mathbf{B}/4\pi c)/\rho\mathbf{V} = {V_A}^2/c^2 \; . \tag{27}$$

The main result of this Letter, that under the usual conditions of ${V_A}^2/c^2 \ll 1$ an electric field does not produce a significant plasma bulk flow whereas a flow does produce an electric field, is thus simply a consequence of momentum conservation plus the fact that the linear momentum in the electromagnetic field is very small compared to that in the plasma bulk flow.

Discussion

Although the MHD relation between the electric field and the plasma bulk flow treats both quantities on an equal footing, they can be distinguished, as shown in this Letter, by positing an initial state with only one of the two present and then using the basic equations to follow the subsequent development of both. This method, based strictly on physics with no reference to any philosophical or choice-of-paradigm considerations, unambiguously identifies one of the two as producing the other. As long as the inertia of the plasma is dominated by the rest mass of the plasma particles and not by the relativistic energy-equivalent mass of the magnetic field (that is the significance of the relation ${V_A}^2/c^2 \ll 1$), flows produce electric fields, but electric fields do not produce flows, in a precisely defined sense: if one starts with plasma flow and no electric field, the flow continues and the electric field appears (with the mean value required by MHD) on a time scale defined essentially by the plasma frequency, whereas if one starts with an electric field and no plasma flow, the electric field simply dissolves into plasma waves (with nearly zero mean) and no appreciable flow appears. The reason for this is simple: bulk flow carries linear momentum and thus can be produced only by adding linear momentum to the plasma, which is done by stresses acting on the plasma; adding the momentum density of the electromagnetic field, the sole contribution from the mere presence of the electric field, has a negligible effect if ${V_A}^2/c^2 \ll 1$. (If the electric field is externally applied and maintained, e.g. by a voltage on capacitor plates immersed in the plasma, the flow obviously has been produced not by the electric field itself but by the Lorentz force of the currents that had to be supplied in order to offset the polarization of the plasma and maintain the plate voltage.)

One implication of the results reported here is that several expressions commonly used in discussions of the magnetosphere, e.g. "...electric fields give rise to bursty flows..." or "...magnetospheric convection is driven by an electric field penetrating in from the solar wind..." (some problems with the latter have been discussed by *Parker* [1996]) are inappropriate and distort the underlying physics. Admittedly, such expressions are often used superfluously, in contexts where only the association and not the causal connection of the electric field and the plasma flow is meant, and may therefore be judged inaccurate but harmless. Where the question of what drives what arises, however, one must be precise. The electric fields are consequences of the flows; to explain the flows themselves, stress imbalances and resultant accelerations must be looked for.

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