Elements of area in $\mathbb{R}^3$

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"We are not moved by things, but by the views we take of them."
Epictitus

Given a region S of a surface in $\mathbb{R}^3$ parameterized by

$$f(u,v) = \begin{pmatrix} f^1(u,v) \\ f^2(u,v) \\ f^3(u,v) \end{pmatrix},$$

we find its area using the integral

$$\iint_{f^{-1}(S)} \|f_u \times f_v\| \; du\, dv.$$

The integrand

$$\|f_u \times f_v\| = \sqrt{\|f_u\|^2 \|f_v\|^2 - (f_u \cdot f_v)^2}$$

yields the area of a parallelogram spanned by the vector fields f_u and f_v at a point on the surface with coordinates (u,v). There must be a better way to consider this argument.

Taking the dot product as a Riemannian metric on $T\mathbb{R}^3$, we compute:

$$\begin{aligned}
&f^*(dx \otimes dx + dy \otimes dy + dz \otimes dz) \\
&\quad = f^*(dx \otimes dx) + f^*(dy \otimes dy) + f^*(dz \otimes dz) \\
&\quad = d(f^*x) \otimes d(f^*x) + d(f^*y) \otimes d(f^*y) + d(f^*z) \otimes d(f^*z) \\
&\quad = d(x \circ f) \otimes d(x \circ f) + d(y \circ f) \otimes d(y \circ f) + d(z \circ f) \otimes d(z \circ f) \\
&\quad = df^1 \otimes df^1 + df^2 \otimes df^2 + df^3 \otimes df^3 \\
&\quad = \left(\frac{\partial f^1}{\partial u} du + \frac{\partial f^1}{\partial v} dv\right) \otimes \left(\frac{\partial f^1}{\partial u} du + \frac{\partial f^1}{\partial v} dv\right) \\
&\qquad + \left(\frac{\partial f^2}{\partial u} du + \frac{\partial f^2}{\partial v} dv\right) \otimes \left(\frac{\partial f^2}{\partial u} du + \frac{\partial f^2}{\partial v} dv\right) \\
&\qquad + \left(\frac{\partial f^3}{\partial u} du + \frac{\partial f^3}{\partial v} dv\right) \otimes \left(\frac{\partial f^3}{\partial u} du + \frac{\partial f^3}{\partial v} dv\right) \\
&\quad = \left(\left(\frac{\partial f^1}{\partial u}\right)^2 + \left(\frac{\partial f^2}{\partial u}\right)^2 + \left(\frac{\partial f^3}{\partial u}\right)^2\right) du \otimes du \\
&\qquad + \left(\frac{\partial f^1}{\partial u}\frac{\partial f^1}{\partial v} + \frac{\partial f^2}{\partial u}\frac{\partial f^2}{\partial v} + \frac{\partial f^3}{\partial u}\frac{\partial f^3}{\partial v}\right)(du \otimes dv + dv \otimes du) \\
&\qquad + \left(\left(\frac{\partial f^1}{\partial v}\right)^2 + \left(\frac{\partial f^2}{\partial v}\right)^2 + \left(\frac{\partial f^3}{\partial v}\right)^2\right) dv \otimes dv.
\end{aligned}$$

In other words, the matrix of the metric g_{ij} is

$$\begin{pmatrix} f_u \cdot f_u & f_u \cdot f_v \\ f_u \cdot f_v & f_v \cdot f_v \end{pmatrix}$$

in (u, v) coordinates, where $\cdot$ is just the usual dot product in $\mathbb{R}^3$. We also have

$$\sqrt{\det(g_{ij})} = \|f_u \times f_v\|$$

and

$$\sqrt{f^*\langle v, v\rangle f^*\langle w, w\rangle - (f^*\langle v, w\rangle)^2} = \sqrt{\det(g_{ij})}\sqrt{\|v\|^2\,\|w\|^2 - (v \cdot w)^2}.$$

This is no coincidence — just the chain rule, really, and a bit of linear algebra: at a particular point, a second-order covariant tensor field is just a bilinear form on tangent vectors. Given a change of basis matrix $J : V \to W$, and the matrix of a bilinear form G in the basis W, we have

$$G \cong J^\top G J \quad \text{so} \quad \det(g_{ij}) = \det(J^\top G J) = \det(G)\det{}^2(J).$$

For the dot product, the matrix G is the identity, so this is just $\det^2(J)$.