order differential terms, which leaves a finite and interesting result

$$\tau_{xy} \approx \tau_{yx} \tag{4.40}$$

Had we summed moments about axes parallel to y or x, we would have obtained exactly analogous results

$$\tau_{xz} \approx \tau_{zx} \qquad \tau_{yz} \approx \tau_{zy} \tag{4.41}$$

There is *no* differential angular-momentum equation. Application of the integral theorem to a differential element gives the result, well known to students of stress analysis, that the shear stresses are symmetric: $\tau_{ij} = \tau_{ji}$. This is the only result of this section.[5] There is no differential equation to remember, which leaves room in your brain for the next topic, the differential energy equation.

4.5 The Differential Equation of Energy[6]

We are now so used to this type of derivation that we can race through the energy equation at a bewildering pace. The appropriate integral relation for the fixed control volume of Fig. 4.1 is

$$\dot{Q} - \dot{W}_s - \dot{W}_v = \frac{\partial}{\partial t}\left(\int_{CV} e\rho \, d\mathscr{V}\right) + \int_{CS}\left(e + \frac{p}{\rho}\right)\rho(\mathbf{V}\cdot\mathbf{n})\, dA \tag{3.63}$$

where $\dot{W}_s = 0$ because there can be no infinitesimal shaft protruding into the control volume. By analogy with Eq. (4.20), the right-hand side becomes, for this tiny element,

$$\dot{Q} - \dot{W}_v = \left[\frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x}(\rho u\zeta) + \frac{\partial}{\partial y}(\rho v\zeta) + \frac{\partial}{\partial z}(\rho w\zeta)\right] dx\, dy\, dz \tag{4.42}$$

where $\zeta = e + p/\rho$. When we use the continuity equation by analogy with Eq. (4.21), this becomes

$$\dot{Q} - \dot{W}_v = \left(\rho \frac{de}{dt} + \mathbf{V}\cdot\nabla p\right) dx\, dy\, dz \tag{4.43}$$

To evaluate $\dot{Q}$, we neglect radiation and consider only heat conduction through the sides of the element. The heat flow by conduction follows Fourier's law from Chap. 1

$$\mathbf{q} = -k\,\nabla T \tag{1.29a}$$

where k is the coefficient of thermal conductivity of the fluid. Figure 4.6 shows the heat flow passing through the x faces, the y and z heat flows being omitted for clarity. We can list these six heat-flux terms:

Faces	Inlet heat flux	Outlet heat flux
x	$q_x\, dy\, dz$	$\left[q_x + \frac{\partial}{\partial x}(q_x)\, dx\right] dy\, dz$
y	$q_y\, dx\, dz$	$\left[q_y + \frac{\partial}{\partial y}(q_y)\, dy\right] dx\, dz$
z	$q_z\, dx\, dy$	$\left[q_z + \frac{\partial}{\partial z}(q_z)\, dz\right] dx\, dy$

[5] We are neglecting the possibility of a finite *couple* being applied to the element by some powerful external force field. See, e.g., Ref. 6, p. 217.

[6] This section may be omitted without loss of continuity.

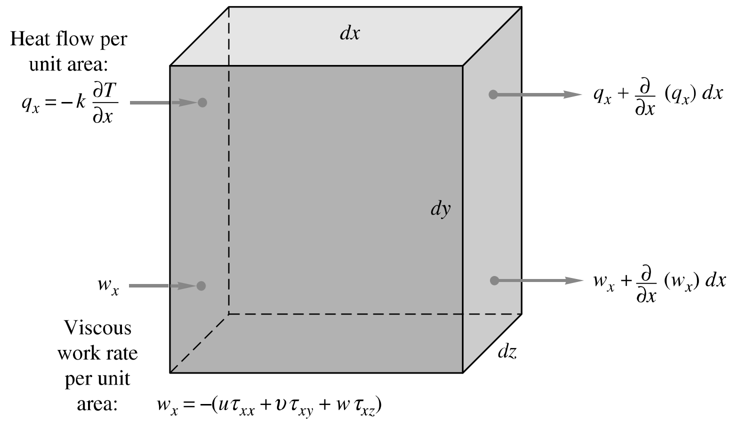

Fig. 4.6 Elemental cartesian control volume showing heat-flow and viscous-work-rate terms in the x direction.

By adding the inlet terms and subtracting the outlet terms, we obtain the net heat added to the element

$$\dot{Q} = -\left[\frac{\partial}{\partial x}(q_x) + \frac{\partial}{\partial y}(q_y) + \frac{\partial}{\partial z}(q_z)\right] dx\,dy\,dz = -\nabla \cdot \mathbf{q}\,dx\,dy\,dz \qquad (4.44)$$

As expected, the heat flux is proportional to the element volume. Introducing Fourier's law from Eq. (1.29), we have

$$\dot{Q} = \nabla \cdot (k\,\nabla T)\,dx\,dy\,dz \qquad (4.45)$$

The rate of work done by viscous stresses equals the product of the stress component, its corresponding velocity component, and the area of the element face. Figure 4.6 shows the work rate on the left x face is

$$\dot{W}_{\upsilon,\mathrm{LF}} = w_x\,dy\,dz \qquad \text{where } w_x = -(u\tau_{xx} + \upsilon\tau_{xy} + w\tau_{xz}) \qquad (4.46)$$

(where the subscript LF stands for left face) and a slightly different work on the right face due to the gradient in w_x. These work fluxes could be tabulated in exactly the same manner as the heat fluxes in the previous table, with w_x replacing q_x, etc. After outlet terms are subtracted from inlet terms, the net viscous-work rate becomes

$$\begin{aligned}\dot{W}_\upsilon = &-\left[\frac{\partial}{\partial x}(u\tau_{xx} + \upsilon\tau_{xy} + w\tau_{xz}) + \frac{\partial}{\partial y}(u\tau_{yx} + \upsilon\tau_{yy} + w\tau_{yz})\right. \\ &\left. + \frac{\partial}{\partial z}(u\tau_{zx} + \upsilon\tau_{zy} + w\tau_{zz})\right] dx\,dy\,dz \\ = &-\nabla \cdot (\mathbf{V} \cdot \boldsymbol{\tau}_{ij})\,dx\,dy\,dz \qquad (4.47)\end{aligned}$$

We now substitute Eqs. (4.45) and (4.47) into Eq. (4.43) to obtain one form of the differential energy equation

$$\rho\,\frac{de}{dt} + \mathbf{V} \cdot \nabla p = \nabla \cdot (k\,\nabla T) + \nabla \cdot (\mathbf{V} \cdot \boldsymbol{\tau}_{ij}) \qquad \text{where } e = \hat{u} + \tfrac{1}{2}V^2 + gz \quad (4.48)$$

A more useful form is obtained if we split up the viscous-work term

$$\nabla \cdot (\mathbf{V} \cdot \boldsymbol{\tau}_{ij}) \equiv \mathbf{V} \cdot (\nabla \cdot \boldsymbol{\tau}_{ij}) + \Phi \qquad (4.49)$$

where Φ is short for the *viscous-dissipation function*.[7] For a newtonian incompressible viscous fluid, this function has the form

$$\Phi = \mu\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial \upsilon}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial \upsilon}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial \upsilon}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2\right] \tag{4.50}$$

Since all terms are quadratic, viscous dissipation is always positive, so that a viscous flow always tends to lose its available energy due to dissipation, in accordance with the second law of thermodynamics.

Now substitute Eq. (4.49) into Eq. (4.48), using the linear-momentum equation (4.32) to eliminate $\nabla \cdot \boldsymbol{\tau}_{ij}$. This will cause the kinetic and potential energies to cancel, leaving a more customary form of the general differential energy equation

$$\rho \frac{d\hat{u}}{dt} + p(\nabla \cdot \mathbf{V}) = \nabla \cdot (k\, \nabla T) + \Phi \tag{4.51}$$

This equation is valid for a newtonian fluid under very general conditions of unsteady, compressible, viscous, heat-conducting flow, except that it neglects radiation heat transfer and internal *sources* of heat that might occur during a chemical or nuclear reaction.

Equation (4.51) is too difficult to analyze except on a digital computer [1]. It is customary to make the following approximations:

$$d\hat{u} \approx c_\upsilon\, dT \qquad c_\upsilon,\ \mu,\ k,\ \rho \approx \text{const} \tag{4.52}$$

Equation (4.51) then takes the simpler form

$$\rho c_\upsilon \frac{dT}{dt} = k\nabla^2 T + \Phi \tag{4.53}$$

which involves temperature T as the sole primary variable plus velocity as a secondary variable through the total time-derivative operator

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + \upsilon\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} \tag{4.54}$$

A great many interesting solutions to Eq. (4.53) are known for various flow conditions, and extended treatments are given in advanced books on viscous flow [4, 5] and books on heat transfer [7, 8].

One well-known special case of Eq. (4.53) occurs when the fluid is at rest or has negligible velocity, where the dissipation Φ and convective terms become negligible

$$\rho c_\upsilon \frac{\partial T}{\partial t} = k\, \nabla^2 T \tag{4.55}$$

This is called the *heat-conduction equation* in applied mathematics and is valid for solids and fluids at rest. The solution to Eq. (4.55) for various conditions is a large part of courses and books on heat transfer.

This completes the derivation of the basic differential equations of fluid motion.

[7]For further details, see, e.g., Ref. 5, p. 72.