Energy flux in a conducting wire

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March 30, 2022

1 Introduction

I investigate the claim that energy doesn't flow in wires, but in the space around them. The claim comes from focusing on the Poynting vector and by assuming that the conducting wire of a circuit has no resistance. However, it turns out that there is energy "flowing" in the wire in all cases, including when the resistivity vanishes. The argument comes from thermodynamics. The only assumptions involved are that the first law of thermodynamics hold (conservation of energy), and that there is a uniform current density in the wire (holds when thermoelectric effects are neglected, and a DC is used, or low frequency AC. The claim would still hold if those effects would be taken into account, but the math would be more tedious to deal with.). The uniform $\vec{J}_e$ condition is reasonable, and arises for a linear charge distribution at the wire's surface, which is a good approximation to what happens when a power source is connected to a circuit.

1.1 Poynting vector route

The Poynting vector $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$ is the energy flux coming from the EM fields. From now and on, I will consider the region consisting of the interior of the wire. The continuity equation applied to the EM energy density u yields

$$\frac{\partial u}{\partial t} = -\nabla\cdot\vec{S} - \vec{J}_e\cdot\vec{E}.$$

So that in steady state, $\nabla\cdot\vec{S} = -\vec{J}_e\cdot\vec{E}$. We can rewrite $\vec{E}$ using Ohm's law $\vec{J}_e = -\sigma\nabla\overline{\mu}$ where $\overline{\mu}$ is the electrochemical potential. By doing so, one finds that

$$\nabla\cdot\vec{S} = -\rho|\vec{J}_e|^2. \tag{1}$$

From this expression, it is evident that the Poynting vector does not catch the whole energy involved in the system, because in steady state, the divergence of the (total) energy flux must vanish. Anyway, let's ignore this for now. In the wire, $\vec{E}$ points along the wire, because $\vec{E}$ is directly proportional to $\vec{J}_e$, and it is constant everywhere in the wire. $\vec{B}$ on the other hand has direction $\hat{\theta}$ (the unit vector in cylindrical coordinates), and goes like r because it is proportional to the current enclosed by a cross section of radius r divided by the perimeter of this cross section. Mathematically, $\vec{B} = \frac{\mu_0 I_0 r}{2\pi R^2}\hat{\theta}$. This means the magnetic field vanishes at the center of the wire, and so does the Poynting vector. The direction of the Poynting vector is radially inward the wire, i.e. it points towards the center of the wire ($-\hat{r}$), and its magnitude goes like r, i.e. its magnitude decreases from the surface of the wire towards its center. In the case of zero resistivity (or infinite conductivity), the electric field vanishes inside the wire, and so does the Poynting vector. From all of this, people have claimed that the energy does not flow in the wire (and in particular that it doesn't flow in the direction of the wire, and that it flows outside the wire, since Poynting vector does not vanish there). It turns out this is not correct, as will be showned in the next section.

1.2 Thermodynamics route

In the wire (as a thermodynamics system out of equilibrium due to the current, and as we will see, thermal gradients), $dU = TdS + \overline{\mu}dN$. U is the internal energy and takes into account all forms of energy in the wire. S here is the entropy (to distinguish from the Poynting vector, I would use $|\vec{S}|$ to denote the magnitude of the Poynting vector, if needs to occur.) This means that

$$\vec{J_U} = T\vec{J_S} + \overline{\mu}\vec{J_e} \tag{2}$$

when it comes to fluxes (energies per unit surface area per unit time). $\vec{J_U}$ is the internal energy flux, a quantity defined at every single point in the wire. $\vec{J_S}$ is the entropy flux. In steady state, $\nabla \cdot \vec{J_U} = 0$, i.e. the heat that enters any part of the wire must leave it (no accumulation of energy anywhere). Then, using $\vec{J_S} = \vec{J_Q}/T$ where $\vec{J_Q}$ is the heat flux, given by Fourier's law (the analog of Ohm's law for the thermal flux rather than electric flux): $\vec{J_Q} = -\kappa\nabla T$, one finds the heat equation

$$\nabla \cdot (-\kappa\nabla T) - \rho|\vec{J_e}|^2 = 0 \tag{3}$$

which is the equation temperature must satisfy in the material (regardless of the boundary conditions that we choose to apply). This equation tells us that in any volume element, the quantity of energy generated by Joule heat must be conducted away via heat conduction, in steady state. It also shows that as long as there is a Joule heat, $\nabla T \neq \vec{0}$, i.e. there must exist a thermal gradient at every single point in the wire. It is impossible to keep the wire in isothermal conditions as long as Joule heat is present. It turns out that the thermal gradient must points inward the wire, in the radial direction. Furthermore, due to the symmetry, it must vanish at the center of the wire.

Looking back at equations 1 and 3, we can see that $\nabla \cdot (-\kappa\nabla T) + \nabla \cdot \vec{S} = 0$, from which we can infer that $\vec{S} = \kappa\nabla T$. As a checkup, we see that these expressions share the same direction, and magnitude (in particular they vanish at the center of the wire). Their magnitude is maximum at the boundaries of the wire. To complete the proof, see appendix A. Solving the heat equation in the particular case of the temperature kept fixed at the surface of the wire yields $T(r) = \frac{\rho|\vec{J_e}|^2}{4\kappa}(R^2 - r^2) + T_0$, which implies $\nabla T = -\frac{\rho|\vec{J_e}|^2 r}{2\kappa}\hat{r}$.

1.2.1 Limit when $\rho \to 0$

In the limit of zero resistivity (perfect conductor), Joule heat goes away, as well as the Poynting vector (which remains non zero outside the wire). In that case, there is no thermal gradient in the wire. But there is still the $\overline{\mu}\vec{J_e}$ term appearing in the expression of the internal energy flux. In fact, the internal energy flux is precisely equal to that quantity. Therefore, energy "flows" inside the wire, even though the conductor is perfect (no electrical resistance).

2 Expression of the energy flux in the wire

We have all the elements to write down the expression for the energy flux in the wire, for both the case of finite and vanishing resistivity. In the case of finite ρ, the condition that the current density is uniform implies that $\overline{\mu} = \overline{\mu_0} + \mu_1 z$, in other words that the potential drop is linear along the wire. We can go a little bit further, $\nabla\overline{\mu} = \mu_1\hat{z} = -\rho|\vec{J_e}|\hat{z}$

Referring back to eq 2, we finally reach

$$\boxed{\vec{J_U} = \frac{\rho|\vec{J_e}|^2 r}{2}\hat{r} + (\overline{\mu_0} - \rho|\vec{J_e}|z)|\vec{J_e}|\hat{z}} \tag{4}$$

This expression can be used to sketch the total energy flux in the wire. In the case of zero resistivity, only a constant $\hat{z}$ component survives. As the resistivity is increased, there appear a radial component whose magnitude increases the further we are from the center, as well as a decrease/increase in the longitudinal component, which reflects the potential drop due to the resistitivity.

3 (Interesting) questions that arise

- What happens in the case of a superconductor? Is that any different than in the case of a perfect conductor? Intuitive answer: probably not different, unfortunately. If thermoelectricity is taken into account, we would still fall back to the same case since the Seebeck coefficient of a SC vanishes.
- How does the analysis change when thermoelectricity is taken into account? Short answer: do the math, start by modifying the Ohm and Fourier's laws to their generalization in thermoelectricity.
- What fraction of the energy is conducted in the wire compared to the one outside the wire, in function of the value of the resistivity? Good question...

- And all cross questions, i.e. considering thermoelectricity, etc.
- What power must the power source provide to keep the steady state? Answer: the question of the power is a bit subtle. First, the Poynting vector makes up for the charge distribution on the surface of the wire. These charges create the uniform current density in the wire, which is associated to an energy flux in the wire, and along the wire, everywhere inside the wire. When the resistivity is non zero, a radial thermal flux appears in the wire, as it must evacuate the generated Joule heat. It turns out that this outgoing thermal flux is equal to minus the Poynting vector. It is not the total energy density flux of the system, unlike what is sometimes claimed.

  The power source has to spend energy to create the electrical current in the wire, even in the case of vanishing resistivity. In that particular case, the energy required to set up the current is equal to $\int_V \overline{\mu}\vec{J}_e \cdot d\vec{l}$ (where $\overline{\mu}$ is constant through the wire, which is not the case of non zero resistivity.). This means that, if we compare two similar wires, but one has a current and the other doesn't have it, the one with the current has more energy inside it than the other. When the current stops, and if the wire has a null resistivity, the power may be radiated away (although I am not sure, but that would make sense). In any case, that extra energy has to go somewhere, and cannot be dissipated in the wire if it has no resistivity.

  To maintain the steady state, the power source must give up what is lost as Joule heat, i.e. the heat that is conducted away.
- How to reconciliate the 2 views? For example, what generates Joule heat (if the question makes any sense)? From the Poynting vector way, it seems like it's the divergence of that vector that does it. However, from the thermodynamics point of view, it's the divergence of the $\overline{\mu}\vec{J}_e$ terms that does it. These divergences are certainly equal, but the energy fluxes are not equal, they do not point in the same direction. Joule heat comes from the interaction of the electrons with the phonons, I think. In any case, ρ must not be null for it to exist.

A Proof that the Poynting vector is equal to the radially conducted heat

I use cylindrical coordinates (r, θ, z). $\vec{S} = |\vec{S}|\hat{r}$. The divergence in cylindrical coordinate yields $\nabla \cdot \vec{A} = \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + ...$ So that $\nabla \cdot \vec{S} = \frac{1}{r}\left(S_r + r\frac{\partial S}{\partial r}\right)$. But we know that $S_r \propto r$, so that $S_r = ar$, and so $\nabla \cdot \vec{S} =$ constant, and that makes sense, since we know that it should give Joule heat (which does not depend on r). If we look at the expression for ∇T obtained by solving the heat equation, from it we can conclude that $\kappa \nabla T \propto r$, which makes sense, as this quantity is equal to the Poynting vector which we know is proportional to r.

There is a condition that must hold if $\vec{S} = \kappa \nabla T$:

$$-\frac{\rho|\vec{J}_e|^2 r\hat{r}}{2} = \frac{1}{\mu_0}\vec{E} \times \vec{B} = \mu_0 \rho |\vec{J}_e| \hat{z} \times \frac{\mu_0 I_0 r}{2\pi R^2}\hat{\theta}$$

and indeed, it holds (do the math if you want.). This fully justifies the equality.