Short Essay: Ideals of Direct Products

Let $R \times S$ be a direct product of rings. Then any ideal of $R \times S$ is $I \times J$, for some ideals I, J of R, S, respectively.

This statement is false.

Counterexample. Let $\mathbf{Z}'$ be a ring of integers with usual addition but zero multiplication, i.e., $ab = 0\ \forall a, b \in \mathbf{Z}'$. $\mathbf{Z}' \times \mathbf{Z}'$ is a ring. Then let $K = \{(3n, 3n) | n \in \mathbf{Z}\}$. Clearly, K is an ideal of $\mathbf{Z}' \times \mathbf{Z}'$, but is not of the form $A \times B$, where $A \subset \mathbf{Z}'$ and $B \subset \mathbf{Z}'$.

As one can see, a problem occurs when we try to show that any ideal K of $R \times S$ is of the form $A \times B$, $A \subset R, B \subset S$. To show that this is so, we must show that $(a, m), (b, n) \in K$ iff $(a, n), (b, m) \in K$. But anyone who tries will find this quite difficult to prove, and as the counterexample shows, also quite impossible.

Proposition. Let $R \times S$ be a direct product of rings, where R and S are rings with unity. Then any ideal of $R \times S$ is $I \times J$, for some ideals I, J of R, S, respectively.

Proposed solution. Let K be an ideal of $R \times S$. We show that $(a, m), (b, n) \in K$ iff $(a, n), (b, m) \in K$. $(a, m) \cdot (1, 0) = (a, 0) \in K$ and $(b, n) \cdot (0, 1) = (0, n) \in K$. So, $(a, n) \in K$. The rest follows similarly. Thus, K is of the form $A \times B$, where $A \subset R$ and $B \subset S$. Now let $I = \{r | (r, 0) \in K\}$, $J = \{s | (0, s) \in K\}$. Then I and J are ideals of R and S, as one can show without too much difficulty. Now it follows that $K = I \times J$.

Example. Find all the ideals of $\mathbf{Z} \times \mathbf{Q}$

Proposed solution. First we note that both $\mathbf{Z}$ and $\mathbf{Q}$ have unity. Thus, an ideal of the direct product $\mathbf{Z} \times \mathbf{Q}$ is of the form $A \times B$, where A is an ideal of $\mathbf{Z}$, and B an ideal of $\mathbf{Q}$. Then since $\mathbf{Q}$ is a field and thus only has trivial ideals, all ideals of $\mathbf{Z} \times \mathbf{Q}$ must be of the form $\mathbf{Z} \times \{0\}$ or $\mathbf{Z} \times \mathbf{Q}$. Now, all ideals are subrings, and all subrings of $\mathbf{Z}$ are also its ideals. Since any subring of $\mathbf{Z}$ is $n\mathbf{Z}$ for some $n \in \mathbf{Z}$, any ideal of $\mathbf{Z}$ must be $n\mathbf{Z}$. Hence I is an ideal of $\mathbf{Z} \times \mathbf{Q}$ iff $I = n\mathbf{Z} \times \{0\}$ or $I = n\mathbf{Z} \times \mathbf{Q}$.