Question

Consider an inertial frame S with coordinates $x^\mu = (t, x, y, z)$ and a frame S' with coordinates $x^{\mu'}$ related to S by a boost with velocity parameter v along the y-axis. Imagine we have a wall at rest in S', lying along the line x' = -y'. From the point of view of S, what is the relationship between the incident angle of a ball hitting the wall (travelling in the x-y plane) and the reflected angle? What about the velocity before and after?

Answer / Part 1, the angles

By George Keeling

The situation is shown in the diagram below. To begin with I thought that the angles could be calculated without knowing the ball's velocities. Axolotl's answer rally made me reconsider that. I was led on a wild goose chase by our cunning question setter.

From the S' point of view S has a boost of -v along the Y axis. The angle of incidence of the ball is α'. The path of the ball in S' is also projected through the wall so that we can easily see the angle of the ball's line to the X' axis is $\alpha' + \pi/4$. The wall is moving upwards in S and the ball, before bouncing, is moving upwards even faster.

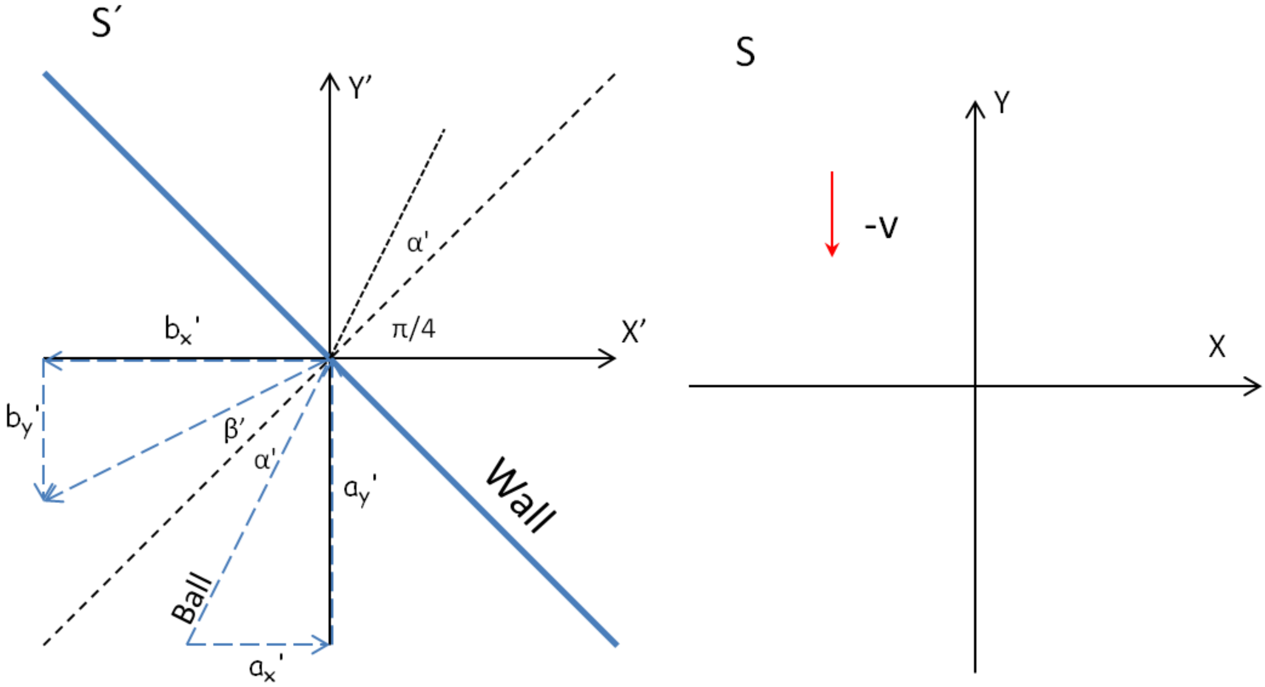

Fig. 1

In this exercise we make use of the trigonometric identities

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right), \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right),\ \tan^{-1}\left(\frac{m}{n}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{n}{m}\right)$$

The incoming velocity of the ball is (a'_x, a'_y), the reflected angle is $\beta' = \alpha'$. By symmetry, the incoming and outgoing velocities are related by

$$a'_x = -b'_y, \qquad a'_y = -b'_x \tag{1}$$

From the geometry

$$a'_x = a'\cos(\alpha' + \pi/4)\ , \qquad a'_y = a'\sin(\alpha' + \pi/4) \tag{2}$$

From equation (1.35) in the book, clearly

$$x = x' \tag{3}$$

And

$$y = \frac{1}{\sqrt{1-v^2}}(y' + vt') \tag{4}$$

$$t = \frac{1}{\sqrt{1-v^2}}(t' + vy') \tag{5}$$

[For our purposes v is always > 0.]

The equation of a straight line in S' is

$$y' = m'x' + k'$$

where m' is the gradient and k' is where the line crosses the Y' axis.

From (1) and (3), in S, the same straight line in S has the equation

$$y = xm'\sqrt{1-v^2} + k'\sqrt{1-v^2} + vt \tag{7}$$

we are only considering the line at an instant (in our case, the event when the ball hits the wall) so the vt part of (7) is constant.

We now have a formula for how the gradient of a line transforms:

$$m = m'\sqrt{1-v^2} \tag{8}$$

The gradient of the wall in S is

$$m = -\sqrt{1-v^2} \tag{8a}$$

So the wall will rotate counter-clockwise as v increases. This is what we would expect from length contraction (thinking of the wall as lots of little steps). The gradient of a perpendicular to a line is the negative reciprocal, so the angle of a perpendicular to the wall above the X-axis is

$$w_\perp = \tan^{-1}\left(1/\sqrt{1-v^2}\right) = \tan^{-1}\gamma \tag{8b}$$

where we have our usual definition of γ.

In my first attempt at this problem, I assumed that the lines followed by the ball could be transformed in the same way and I spent too long chasing this wild goose. Sadly that was wrong as I realised when I read Axolotl's answer. The second part of the question must really be answered before the first.

We can get a formula for straight line, uniform velocities in S in terms of velocities in S' as follows. The velocities are a, a' with components a_x , a_y , a'_x , a'_y .

From (4) and (5)

$$a_x = \frac{\Delta x}{\Delta t} = \frac{\Delta x'}{\gamma(\Delta t' - v\Delta y')} = \frac{a'_x}{\gamma\left(1 + va'_y\right)} \tag{9}$$

$$a_y = \frac{\Delta y}{\Delta t} = \frac{\Delta y' + v\Delta t'}{\Delta t' + v\Delta y'} = \frac{a'_y + v}{1 + va'_y}$$

Therefore

$$\boxed{a^2 = \frac{a'^2_x(1 - v^2) + (a'_y + v)^2}{(1 + va'_y)^2}} \qquad (10)$$

(9) is a perfectly general velocity transformation equation and works just as well for b_x, b_y, the outgoing velocity components. We can use these to calculate the gradients m_{in}, m_{out} of the incoming and outgoing ball in S and the incident and reflected angles α and β.

$$m_{in} = \frac{a_y}{a_x} = \frac{(a'_y + v)\gamma}{a'_x}$$
$$m_{out} = \frac{b_y}{b_x} = \frac{(b'_y + v)\gamma}{b'_x} = \frac{(a'_x - v)\gamma}{a'_y} \qquad (11)$$

We have used (1) in the m_{out} equation.
We can immediately see that α and β depend on the boost, the angle of incidence and the ball's velocity. It's quite complicated. By simple geometry, we have

$$\alpha = \tan^{-1} m_{in} - w_{\perp}$$

Using (11) and (8b) and (3) this becomes

$$\alpha = \tan^{-1}\left(\frac{(a' \sin(\alpha' + \pi/4)\ + v)\gamma}{a' \cos(\alpha' + \pi/4)}\right) - \tan^{-1}\gamma$$

Similarly

$$\beta = \tan^{-1}\gamma - \tan^{-1}\left(\frac{(a' \cos(\alpha' + \pi/4) - v)\gamma}{a' \sin(\alpha' + \pi/4)}\right) \qquad (12)$$

These have the interesting property that for a large boost and modest angle of incidence in S', the angle of reflection is greater than 90°. For example if $\alpha' = 70°$, $v = 0.5$, $a' = 0.5$ then $\beta = 110°$. The ball seems to pass through the wall! That's more like quantum mechanics than relativity. The situation is illustrated in the diagram on the right below

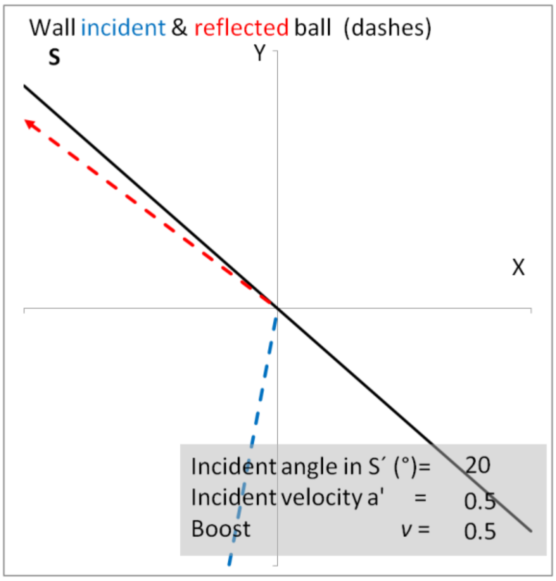

Fig 2: Reflected ball is reflected.

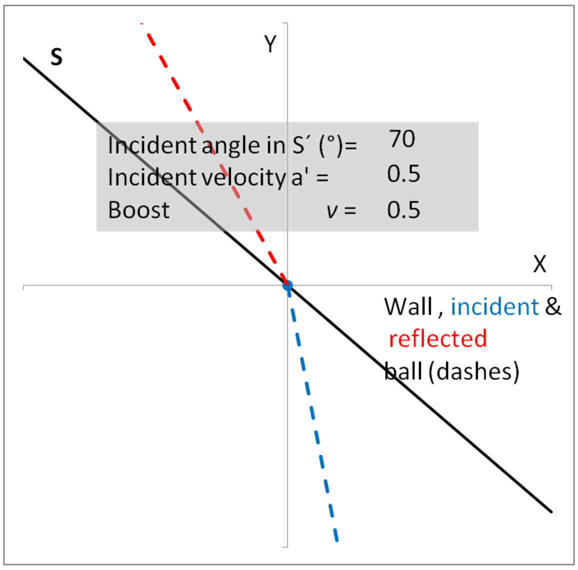

Fig 3: Reflected ball is 'transmitted'

However, the ball is not on one side of the wall in S' and the other in S. The problem is with the diagram and the maths. It is easy to arrange that the ball hits the wall at the spacetime origin in S and S'. Both diagrams show the wall at $t = t' = 0$. However the ball's position is at different times. Only at impact does $t = 0$.

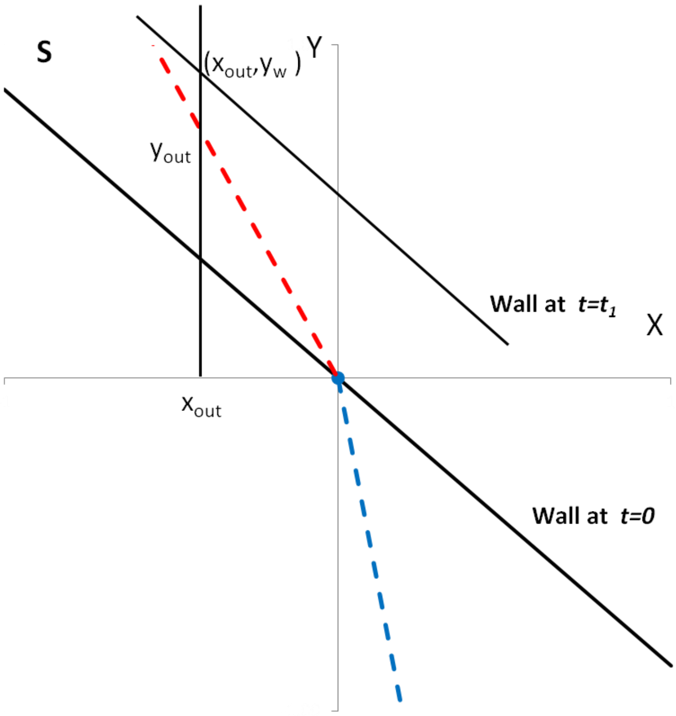

Fig 4: Showing the moving wall.

We need to calculating the position of the wall at a later time $t_1 > 0$ as in Fig 4. At that time the reflected ball is at

$$(x_{out}, y_{out}) = (b_x t_1, b_y t_1) \tag{13}$$

From (7), with $m' = -1$ and $k' = 0$ we have the equation of the wall and we want the coordinates of the point on the wall directly above the ball. These are $(x_{out}\,,\, y_w)$. So

$$y_w = -x_{out}/\gamma + vt_1 \tag{14}$$

or using (13)

$$y_w = -b_x t_1/\gamma + vt_1$$

if $y_w - y_{out} > 0$, then the wall is ahead of the ball and we can breathe a sigh of relief. Using (13) and (14)...

$$y_w - y_{out} = -\frac{b_x t_1}{\gamma} + vt_1 - b_y t_1$$

using the velocity transformations (9)

$$y_w - y_{out} = t_1\left(\frac{-b'_x}{\gamma^2(1 + vb'_y)} + v - \frac{(b'_y + v)}{(1 + vb'_y)}\right)$$

$$= \frac{t_1}{\gamma^2(1 + vb'_y)}\left(-b'_x + v\gamma^2 + v^2\gamma^2 b'_y - b'_y\gamma^2 - v\gamma^2\right)$$

$$= \frac{t_1}{\gamma^2(1 + vb'_y)}\left(-b'_x - b'_y\gamma^2(1 - v^2)\right)$$

and using (1) and remembering what γ^2 is

$$y_w - y_{out} = \frac{t_1}{\gamma^2(1 - va'_x)}(a'_x + a'_y) > 0 \tag{15}$$

t_1 , γ^2 and $(1 - va'_x)$ are always positive. Referring to Fig 1 for $0° \le \alpha' \le 45°$, $a'_x \ge 0$ and $a'_y > 0$, for $45° < \alpha' < 90°$, $a'_y > -a'_x$. So we have proved that the wall is ahead of the ball for all a, v and $0° \le \alpha' < 90°$. We breathe freely. One should check the other three quadrants but I do not have time.

Going back to (13) & (14), at time t, the point on the wall above the ball is at

$$y = mx_{out} + v\frac{x_{out}}{b_x}$$

where m is the gradient of the wall in S. So the line carved out by the point is

$$y = x\left(m + \frac{v}{b_x}\right) \tag{16}$$

This is a very convenient form to be plotted on our graph. We might call this line the 'adjusted wall' and the angle between the reflected ball and the adjusted wall the adjusted angle. Figs 2 & 3 become

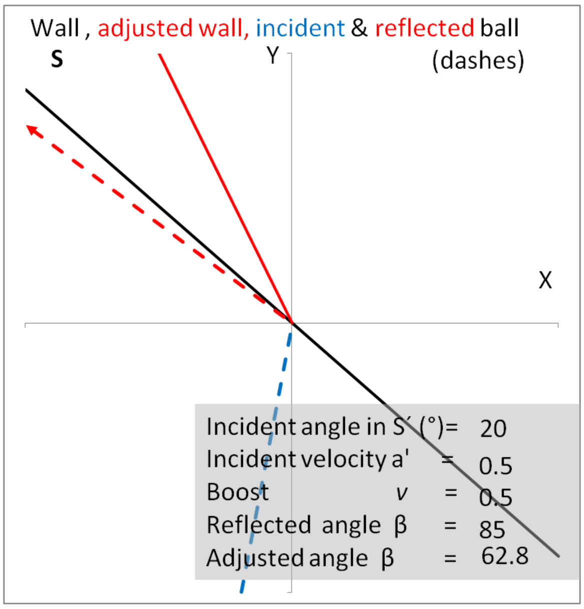

Fig 5

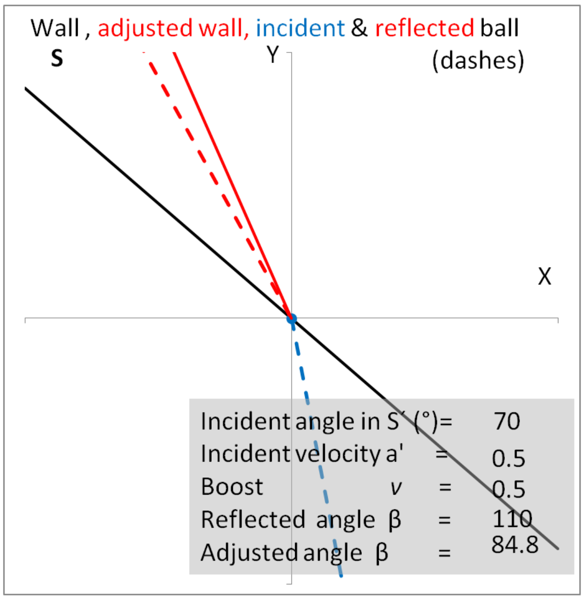

Fig 6

Even for modest angles of incidence in S' $\beta > 90°$. For example with $\alpha' = 35°, \beta = 93°$.

I suppose this shows us that angles of moving objects to each other must be treated with great caution!

We have already calculated the velocities - a few graphs for $v = 0.5$ are shown below. The incident speed in S increases almost linearly with the incident speed in S'. When α' is small the reflected speed decreases for small incident speeds, then catches up.

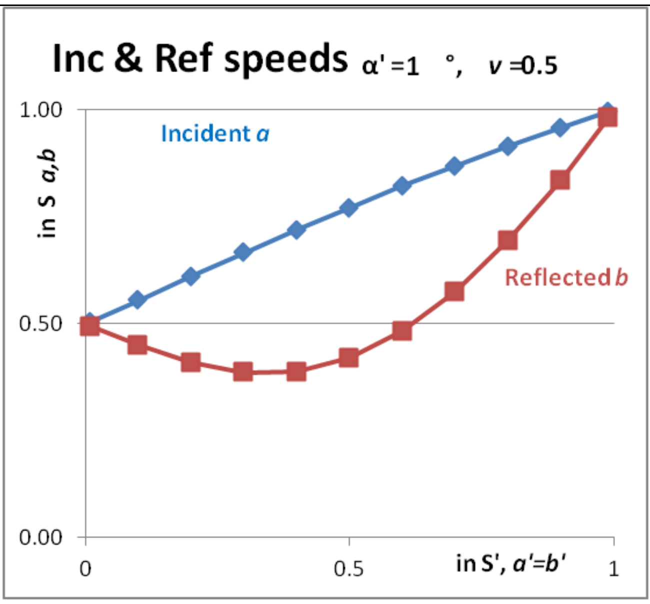

Fig 12

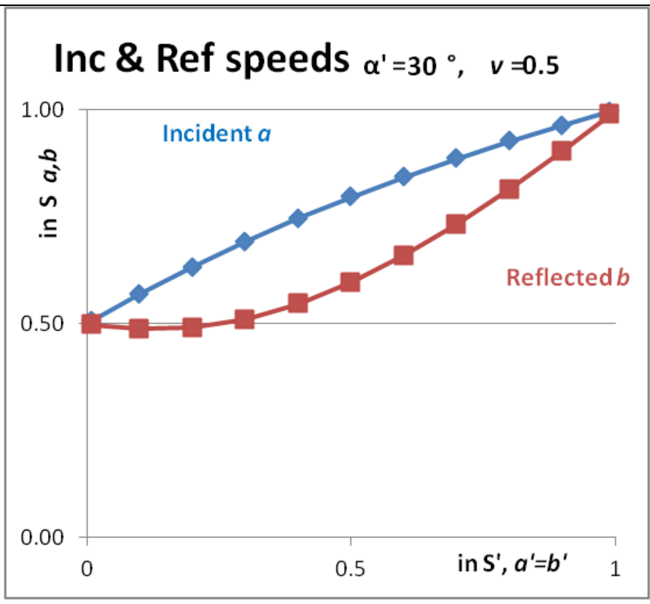

Fig 13

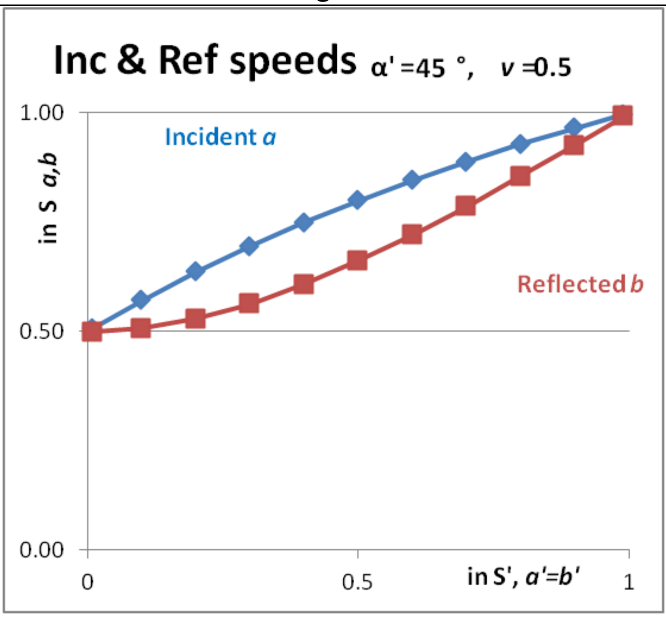

Fig 14

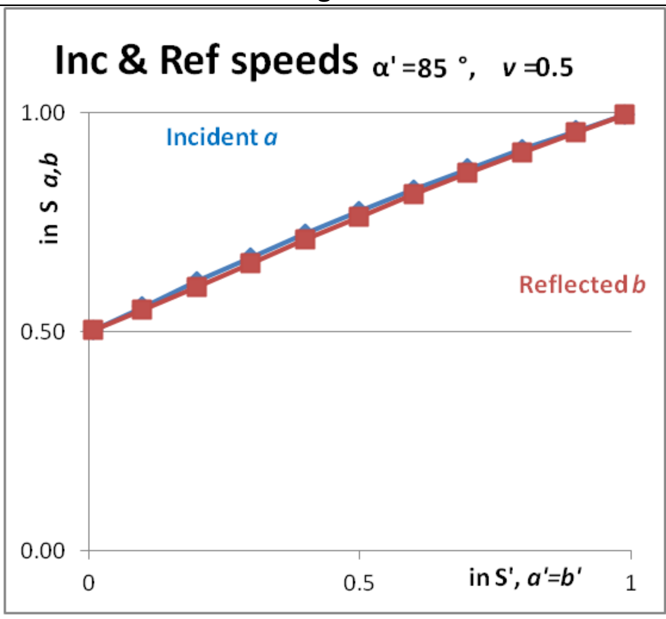

Fig 15

Checking my answers

Possible answer here:
https://www.physicsforums.com/threads/ball-hitting-a-wall-with-relativistic-effects.885190/

That was not very helpful but there is a reference to Lorentz Transformation of velocities in Wikipedia which shows (with a bit of work) that my velocities are correct. It's at
https://en.wikipedia.org/wiki/Lorentz_transformation#Transformation_of_velocities
Starting with the equation

$$\mathbf{u}' = \frac{1}{1-\frac{\mathbf{v}\cdot\mathbf{u}}{c^2}}\left[\frac{\mathbf{u}}{\gamma_{\mathbf{v}}} - \mathbf{v} + \frac{1}{c^2}\frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}}+1}(\mathbf{u}\cdot\mathbf{v})\mathbf{v}\right] \qquad (13)$$

(***u'***, ***u*** are velocities in frames F', F. F′ moves with velocity ***v*** relative to F and $\gamma_v = 1/\sqrt{1-\mathbf{v}.\mathbf{v}/c^2}$.

Proof that (13) gives us (11a).
I was unable to show directly that (13) gives (12). I then re-derived (13) and came up with a different result (15). I was then able to show that it was, after all, the same as (13). I think this is actually simpler:

A very awkward term in (13) was

$$\frac{1}{c^2}\frac{\gamma}{\gamma+1}$$

(I am using γ instead of γ_v here.) This may look a bit off to begin with! We have

$$\frac{1}{c^2}\frac{\gamma}{\gamma+1} = \frac{1}{c^2}\frac{1}{\gamma+1}\left(\frac{\gamma^2}{\gamma}\right) = \frac{1}{c^2}\frac{1}{\gamma+1}\left(\frac{\frac{1}{1-v^2/c^2}}{\gamma}\right)$$

$$= \frac{1}{c^2}\frac{1}{\gamma+1}\left(\frac{\left[\frac{1-1+v^2/c^2}{1-v^2/c^2}\right]c^2}{\gamma v^2}\right)$$

$$= \frac{1}{c^2}\frac{1}{\gamma+1}\left(\frac{\left[\frac{1}{1-v^2/c^2}-1\right]c^2}{\gamma v^2}\right)$$

$$= \frac{1}{c^2}\frac{1}{\gamma+1}\left(\frac{[\gamma^2-1]c^2}{\gamma v^2}\right)$$

Nearly there!

$$= \frac{1}{c^2}\frac{1}{\gamma+1}\left(\frac{[\gamma+1][\gamma-1]c^2}{\gamma v^2}\right)$$

$$= \left(\frac{[\gamma-1]}{\gamma v^2}\right)$$

So we now have the remarkable result:

$$\frac{1}{c^2}\frac{\gamma}{\gamma+1} = \frac{1}{v^2}\left(1-\frac{1}{\gamma}\right) \tag{14}$$

Now substitute (14) into (13) and we get

$$\mathbf{u}' = \frac{1}{1-\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\left[\frac{\mathbf{u}}{\gamma} - \mathbf{v} + \frac{1}{v^2}\left(1-\frac{1}{\gamma}\right)(\mathbf{u}\cdot\mathbf{v})\mathbf{v}\right] \tag{15}$$

This is the equation I got when I re-derived the velocity transformation. I then had prove (14), to show that (15) was indeed the same as (13). It is easy to show that (15) gives (11a) and therefore (12).

In our terms (15) is

$$\boldsymbol{a} = \frac{1}{1-\boldsymbol{v}\cdot\boldsymbol{a}'}\left(\frac{\boldsymbol{a}'}{\gamma} - \boldsymbol{v} + \frac{1}{v^2}\left(1-\frac{1}{\gamma}\right)(\boldsymbol{v}\cdot\boldsymbol{a}')\boldsymbol{v}\right) \tag{16}$$

where $\boldsymbol{v} = (0\,,-v\,,0), v > 0$ and $\boldsymbol{v}\cdot\boldsymbol{a}' = -va'_y$
resolving to co-ordinates we get

$$a_x = \frac{1}{1+va'_y}\left(\frac{a'_x}{\gamma}\right)$$

$$a_y = \frac{1}{1+va'_y}\left(\frac{a'_y}{\gamma} + v + \frac{1}{v^2}\left(1-\frac{1}{\gamma}\right)a'_y v^2\right) \tag{17}$$

simplifying ...

$$a_y = \frac{1}{1+va'_y}(a'_y + v)$$

These are the same as (11a). Squaring all those and adding them up, we get

$$a^2 \equiv |a|^2 = \left(\frac{1}{1-va'_y}\right)^2\left[\left(\frac{a'_x}{\gamma}\right)^2 + \left(a'_y - v\right)^2\right] \tag{18}$$

This is the same as (12), which proves my answer is correct.

Other notes

1) Wikipedia also had fascinating references to the Albert Einstein Reference Archive at
https://www.marxists.org/reference/archive/einstein/works/1910s/relative/
and Einstein on marxists.org at
https://www.marxists.org/reference/archive/einstein/index.htm
They quote him in 1947 saying "I came to America because of the great, great freedom which I heard existed in this country. I made a mistake in selecting America as a land of freedom, a mistake I cannot repair in the balance of my life." He would die eight years later.

2) Petra Axolotl's Blog at
https://petraaxolotl.wordpress.com/chapter-1-special-relativity-and-flat-spacetime/ says

In frame S, the wall has an angle relative to the x-axis $\alpha = tan^{-1}(\sqrt{1-v^2})$. Let the velocity of the ball be w and its incident angle be θ_{in}.

$$w^x = wcos(\frac{\pi}{2} - \alpha + \theta_{in})$$

$$w^y = wsin(\frac{\pi}{2} - \alpha + \theta_{in})$$

In *S'*, we have

$$w^{x'} = \frac{w^x\sqrt{1-v^2}}{1-vw^y}$$

$$w^{y'} = \frac{w^y - v}{1-vw^y}$$

In *S'*, the wall is symmetric with *y*=*x* being its axis of symmetry. Therefore, the reflected velocity *u* satisfies the following.

$$u^{x'} = -w^{y'}$$

$$u^{y'} = -w^{x'}$$

Back in *S*, *u* becomes

$$u^x = \frac{u^{x'}\sqrt{1-v^2}}{1+vu^{y'}}$$

$$u^y = \frac{u^{y'} + v}{1+vu^{y'}}$$

Therefore, the reflected angle θ_{out} and velocity satisfy

$$\theta_{out} = \frac{\pi}{2} - \alpha - tan^{-1}(\frac{u^y}{u^x})$$

$$|u| = \sqrt{u^{x2} + u^{y2}}$$

Unfortunately, there is no simplification in general that makes the result look more elegant.

PA can do without a drawing! I can't. It's shown below. I hope it is clear.

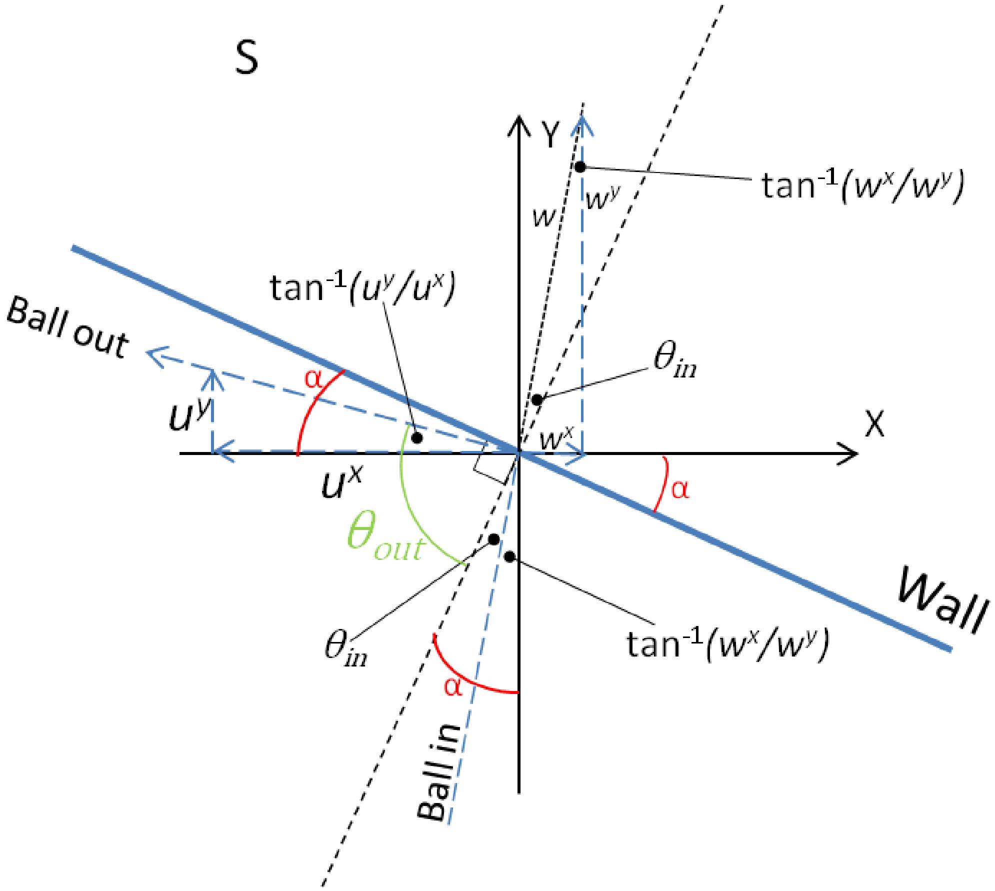

Fig PA1

No evidence is given for PA's very first assertion that $\alpha = tan^{-1}(\sqrt{1-v^2})$. Strictly α is negative. My formula (8a) shows that $\alpha = -\tan^{-1}(\sqrt{1-v^2})$. However it materialises that PA means that α is the angle <u>below</u> the X-axis, as shown in the diagram above.
Looking at the top right quadrant of the diagram, the next two equations are correct. But Axolotl should go further:

$$w^x = w\sin(\alpha - \theta_{in})$$
$$w^y = w\cos(\alpha - \theta_{in})$$

Dividing

$$\tan(\alpha - \theta_{in}) = \frac{w^x}{w^y}$$
$$\theta_{in} = \alpha - \tan^{-1}\left(\frac{w^x}{w^y}\right)$$

This is also obvious from the bottom left quadrant of the diagram, which is a more obvious place to look.
Using the velocity transformation equations,

$$\theta_{in} = \alpha - \tan^{-1}\left(\frac{w^{x\prime}\sqrt{1-v^2}}{w^{y\prime}+v}\right)$$

Expanding α , using our old friend γ and using the $\tan^{-1}$ trigonometric identity twice we get

$$\theta_{in} = \tan^{-1}\left(\frac{1}{\gamma}\right) - \tan^{-1}\left(\frac{w^{x\prime}\sqrt{1-v^2}}{w^{y\prime}+v}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\gamma - \frac{\pi}{2} + \tan^{-1}\left(\frac{(w^{y'} + v)\gamma}{w^{x'}}\right)$$

So

$$\theta_{in} = \tan^{-1}\left(\frac{(w^{y'} + v)\gamma}{w^{x'}}\right) - \tan^{-1}\gamma$$

Which, using (1), is the same as the first part of (12). Hurrah!

The four equations for velocity transformations are the same as my (11a) but Axolotl gives no proof. The relationships between reflected and incident velocities in S' are the same as (1).

PA's second last equation is also obvious by subtracting angles around θ_{out} and the wall. We have

$$\theta_{out} + \tan^{-1}\left(\frac{u^y}{u^x}\right) = \frac{\pi}{2} - \alpha$$

The sign of $\tan^{-1}(u^y/u^x)$ is important here. In the diagram id is negative. If θ_{out} were smaller, it would be positive. This gives us PA's second last

$$\theta_{out} = \frac{\pi}{2} - \alpha - \tan^{-1}\left(\frac{u^y}{u^x}\right)$$

trigonometric identity on the first two terms and the velocity transformations on the third we get

$$\theta_{out} = \tan^{-1}\gamma - \tan^{-1}\left(\frac{(u^{y'} + v)\gamma}{u^{x'}}\right)$$

then the relationship between the velocities in S'

$$\theta_{out} = \tan^{-1}\gamma - \tan^{-1}\left(\frac{(w^{x'} - v)\gamma}{w^{y'}}\right)$$

Which is the same as the second part of my (12). Double hurrah.

PA's last is a commonplace of two dimensional geometry.

I opine that Axolotl is a bit incomplete: θ_{in} was never calculated and the fascinating result that θ_{out} can be greater than 90° was never discovered. However Axolotl did help me get my answer correct - I hope.

3) Links to my resources

Ex 1.01 Wall and Ball.docx

Ex 1.01 Wall and Ball.xlsx

Ex 1.01 Wall and Ball.ppt