Exercise 19.1. DERIVATION OF METRIC FAR OUTSIDE A WEAKLY GRAVITATING BODY

EXERCISE

(a) Derive equation (19.5). [*Hints:* (1) Follow the procedure outlined in the text. (2) When calculating h_{00}, write out explicitly the $n = 0$ and $n = 1$ terms of (19.2), to precision $0(1/r^2)$, and simplify the $n = 0$ term using the identities

$$T^{jk} = \frac{1}{2}(T^{00}x^j x^k)_{,00} + (T^{lj}x^k + T^{lk}x^j)_{,l} - \frac{1}{2}(T^{lm}x^j x^k)_{,lm}, \tag{19.7a}$$

$$T^{ll}x^m = \left(T^{0l}x^l x^m - \frac{1}{2}T^{0m}r^2\right)_{,0} + \left(T^{lk}x^k x^m - \frac{1}{2}T^{lm}r^2\right)_{,l}. \tag{19.7b}$$

(Verify that these identities follow from $T^{\alpha\beta}{}_{,\beta} = 0$.) (3) When calculating h_{0m}, write out explicitly the $n = 0$ term of (19.2), to precision $0(1/r^2)$, and simplify it using the identity

$$T^{0k}x^j + T^{0j}x^k = (T^{00}x^j x^k)_{,0} + (T^{0l}x^j x^k)_{,l}. \tag{19.7c}$$

(Verify that this follows from $T^{\alpha\beta}{}_{,\beta} = 0$.) (4) Simplify h_{00} and h_{0m} by the gauge transformation generated by

$$\begin{aligned}
\xi_0 = {} & \frac{1}{2r}\frac{\partial}{\partial t}\int T^{00\prime}r'^2\, d^3x' + \frac{x^j}{r^3}\int\left(T^{0k\prime}x^{k\prime}x^{j\prime} - \frac{1}{2}T^{0j\prime}r'^2\right)d^3x' \\
& + \int (T_{00}{}' + T_{ll}{}')\left[\frac{x^j x^{j\prime}}{r^2} + \frac{(3x^{j\prime}x^{k\prime} - r'^2\delta_{jk})x^j x^k}{2r^4}\right]d^3x' \\
& + \sum_{n=2}^{\infty}\frac{1}{n!}\frac{\partial^{n-1}}{\partial t^{n-1}}\int (T_{00}{}' + T_{kk}{}')\frac{(r - |\boldsymbol{x} - \boldsymbol{x}'|)^n}{|\boldsymbol{x} - \boldsymbol{x}'|}\, d^3x',
\end{aligned}$$

$$\begin{aligned}
\xi_m = {} & -\frac{2x^j}{r^3}\int T_{00}{}'x^{j\prime}x^{m\prime}\, d^3x' + 4\sum_{n=1}^{\infty}\frac{1}{n!}\frac{\partial^{n-1}}{\partial t^{n-1}}\int T_{0m}{}'\frac{(r - |\boldsymbol{x} - \boldsymbol{x}'|)^n}{|\boldsymbol{x} - \boldsymbol{x}'|}\, d^3x' \\
& + \frac{x^m}{r}\xi_0 - \frac{1}{2}\left(\frac{1}{r}\right)_{,m}\int T_{00}{}'r'^2\, d^3x' - \left(\frac{x^k}{r^2}\right)_{,m}\int\left(T^{0j\prime}x^{j\prime}x^{k\prime} - \frac{1}{2}T^{0k\prime}r'^2\right)d^3x' \\
& - \sum_{n=2}^{\infty}\frac{1}{n!}\frac{\partial^{n-2}}{\partial t^{n-2}}\int (T_{00}{}' + T_{kk}{}')\left[\frac{(r - |\boldsymbol{x} - \boldsymbol{x}'|)^n}{|\boldsymbol{x} - \boldsymbol{x}'|}\right]_{,m} d^3x'
\end{aligned}$$

Here $T_{\mu\nu}{}'$ denotes $T_{\mu\nu}(t - r, \boldsymbol{x}')$.]

(b) Prove that the system's mass and angular momentum are conserved. [*Note:* Because $T^{\alpha\beta}{}_{,\beta} = 0$ (self-gravity has negligible influence), the proof is no different here than in flat spacetime (Chapter 5).]