22.6

Note that $1 < l < m$ is most likely a typo for $1 \leq l < m$. Moreover, the below attempt assumes $\int_{[-1,1]} k^2(u)\mathrm{d}u < \infty$. To obtain a sensible estimation, $h \to 0$ as $n \to \infty$.

1. By linearity of the expectation, identical distribution of $x_1, ..., x_n$, the law of the unconscious statistician and the change of variables $u = (t - x)/h$, where $x = x_i$ for some i in $\{1, ..., n\}$,

$$\begin{aligned}
\mathbb{E}[f_n(t)] &= \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left[\frac{1}{h}k\left(\frac{t-x_i}{h}\right)\right] && (1)\\
&= \mathbb{E}\left[\frac{1}{h}k\left(\frac{t-x}{h}\right)\right] && (2)\\
&= \int_{[-1,1]} \frac{1}{h}k\left(\frac{t-x}{h}\right)f(x)\mathrm{d}x && (3)\\
&= \int_{[-1,1]} \frac{1}{h}k(u)f(t-hu)h\mathrm{d}u && (4)\\
&= \int_{[-1,1]} k(u)f(t-hu)\mathrm{d}u. && (5)
\end{aligned}$$

The integration occurs over $[-1, 1]$ since $\mathrm{supp}(k) = [-1, 1]$ and f is a probability density function. Its support is unspecified, but since it integrates to 1 over $\mathbb{R}$, it is bounded.

2. Let $m > 1$ be fixed. From $f \in C^m$, it follows that

$$f(t-hu) = \sum_{l=0}^{m} \frac{f^{(l)}(t)}{l!}(-hu)^l + o((hu)^m). \qquad (6)$$

$o(g(y))$ is a set (or function) such that $f(y) \in o(g(y))$ (or $f(y) = o(g(y))$) satisfies $\lim_{y\to y_0} f(y)/g(y) = 0$ for y_0 denoting a real number, a complex number or $\pm\infty$[1]. In (6), $y_0 = 0$. From (5) and linearity of integration,

$$\begin{aligned}
\mathbb{E}[f_n(t)] &= \int_{[-1,1]} k(u)\left(\sum_{l=0}^{m}\frac{f^{(l)}(t)}{l!}(-hu)^l + o((hu)^m)\right)\mathrm{d}u && (7)\\
&= \sum_{l=0}^{m}\int_{[-1,1]} k(u)\frac{f^{(l)}(t)}{l!}(-hu)^l\mathrm{d}u + \int_{[-1,1]} k(u)o((hu)^m)\mathrm{d}u. && (8)
\end{aligned}$$

From the given conditions on k, the $l = 0$ term reads

$$\int_{[-1,1]} k(u)f(t)\mathrm{d}u = f(t)\int_{[-1,1]} k(u)\mathrm{d}u = f(t). \qquad (9)$$

The $1 \leq l < m$ terms are

$$\int_{[-1,1]} k(u)\frac{f^{(l)}(t)}{l!}(-hu)^l\mathrm{d}u = \frac{f^{(l)}(t)(-h)^l}{l!}\int_{[-1,1]} k(u)u^l\mathrm{d}u = 0. \qquad (10)$$

Finally, the $l = m$ term is

$$\frac{f^{(m)}(t)(-h)^m}{m!}\int_{[-1,1]} k(u)u^m\mathrm{d}u < \infty. \qquad (11)$$

Regarding the integral with the remainder, note that $o((hu)^m)$ does not specify the remainder; it only specifies that the remainder converges to 0 faster than $(hu)^m$ as $hu \to 0$. h is the free

[1] $o(g_n)$ is a set (or sequence) ... such that $\lim_{n\to\infty} f_n/g_n = 0$.

variable while u is the bounded variable; $hu \to 0$ means $h \to 0$. Thus the remainder satisfies $o((hu)^m) = u^m o(h^m) = o(h^m)$. If the remainder is only a function of h, the estimation of the integral is straightforward. If it also depends on u, then the integral can be estimated if the remainder, viewed as a sequence of functions, converges uniformly to 0[2]. Then

$$\int_{[-1,1]} k(u)o((hu)^m)\mathrm{d}u = \int_{[-1,1]} k(u)o(h^m)\mathrm{d}u = o(h^m). \tag{13}$$

To summarize,

$$\mathbb{E}[f_n(t)] = f(t) + \frac{f^{(m)}(t)(-h)^m}{m!}\int_{[-1,1]} k(u)u^m\mathrm{d}u + o(h^m), \tag{14}$$

and thus

$$\mathbb{E}[f_n(t)] - f(t) = \frac{f^{(m)}(t)(-h)^m}{m!}\int_{[-1,1]} k(u)u^m\mathrm{d}u + o(h^m) = Ah^m + o(h^m), \tag{15}$$

where $A = A(t) = \frac{f^{(m)}(t)(-1)^m}{m!}\int_{[-1,1]} k(u)u^m\mathrm{d}u < \infty$.

3. By independence and the change of variables in part 2,

$$\begin{aligned}
\mathrm{Var}[f_n(t)] &= \mathrm{Var}\left[\frac{1}{n}\sum_{i=1}^{n}\frac{1}{h}k\left(\frac{t-x_i}{h}\right)\right] && (16)\\
&= \frac{1}{n^2h^2}\sum_{i=1}^{n}\mathrm{Var}\left[k\left(\frac{t-x_i}{h}\right)\right] && (17)\\
&= \frac{1}{nh^2}\mathrm{Var}\left[k\left(\frac{t-x}{h}\right)\right] && (18)\\
&= \frac{1}{nh^2}\left(\mathbb{E}\left[k^2\left(\frac{t-x}{h}\right)\right] - \left(\mathbb{E}\left[k\left(\frac{t-x}{h}\right)\right]\right)^2\right) && (19)\\
&= \frac{1}{nh^2}\left(h\int_{[-1,1]} k^2(u)f(t-hu)\mathrm{d}u - \left(h\int_{[-1,1]} k(u)f(t-hu)\mathrm{d}u\right)^2\right) && (20)\\
&= \frac{1}{nh^2}\left(h\int_{[-1,1]} k^2(u)\left(f(t) - (hu)f^{(1)}(t) + o(hu)\right)\mathrm{d}u + O(h^2)\right) && (21)\\
&= \frac{1}{nh^2}\left(hf(t)\int_{[-1,1]} k^2(u)\mathrm{d}u - h^2f^{(1)}(t)\int_{[-1,1]} k^2(u)u\mathrm{d}u + o(h^2) + O(h^2)\right) && (22)\\
&= \frac{1}{nh^2}\left(hf(t)\int_{[-1,1]} k^2(u)\mathrm{d}u + O(h^2)\right) && (23)\\
&= \frac{f(t)}{nh}\int_{[-1,1]} k^2(u)\mathrm{d}u + o\left(\frac{1}{nh}\right), && (24)
\end{aligned}$$

since

- $\int_{[-1,1]} |k^2(u)u|\mathrm{d}u \le \int_{[-1,1]} k^2(u)\mathrm{d}u < \infty$,

[2] If a sequence of functions g_n converges uniformly to a function g over some compact interval I where g_n and g are integrable, then

$$\lim_{n\to\infty}\int_I g_n(u)\mathrm{d}u = \int_I g(u)\mathrm{d}u. \tag{12}$$

- $o(h^2) + O(h^2) = O(h^2)$,
- and $O(h^2)\frac{1}{nh^2} = O(h)\frac{1}{nh} = o(1)\frac{1}{nh} = o\left(\frac{1}{nh}\right)$.

4.

$$\begin{aligned}
\text{mse}[f_n(t)] &= \text{Var}[f_n(t)] + \text{Bias}^2[f_n(t)] && (25)\\
&= \left(\frac{f(t)}{nh}\int_{[-1,1]} k^2(u)\mathrm{d}u + o\left(\frac{1}{nh}\right)\right) + (Ah^m + o(h^m))^2 && (26)\\
&= \left(\frac{f(t)}{nh}\int_{[-1,1]} k^2(u)\mathrm{d}u + o\left(\frac{1}{nh}\right)\right) + \left(A^2h^{2m} + 2Ah^m o(h^m) + o(h^{2m})\right) && (27)\\
&= \left(\frac{f(t)}{nh}\int_{[-1,1]} k^2(u)\mathrm{d}u + o\left(\frac{1}{nh}\right)\right) + \left(A^2h^{2m} + o(h^{2m}) + o(h^{2m})\right) && (28)\\
&= \left(\frac{f(t)}{nh}\int_{[-1,1]} k^2(u)\mathrm{d}u + o\left(\frac{1}{nh}\right)\right) + \left(A^2h^{2m} + o(h^{2m})\right) && (29)\\
&\approx \frac{f(t)}{nh}\int_{[-1,1]} k^2(u)\mathrm{d}u + A^2h^{2m}. && (30)
\end{aligned}$$

5. From the approximation obtained in part 4, it follows that $\text{mse}[f_n(t)]$, for t fixed, has an absolute minimum since $\text{mse}[f_n(t)] \to \infty$ for $h \to 0$ and $h \to \infty$ (h is strictly positive). The absolute minimum is found by differentiating $\text{mse}[f_n(t)]$ and solving for h when the derivative equals 0, that is

$$h = \left(\frac{f(t)\int_{[-1,1]} k^2(u)\mathrm{d}u}{A^2 2mn}\right)^{1/(2m+1)}. \qquad (31)$$

6. Plugging in the value of h obtained in part 5 into the approximation obtained in part 4, one finds that

$$\text{mse}[f_n(t)] \propto n^{-2m/(2m+1)}, \qquad (32)$$

since both $1/nh$ and h^{2m} reduce to $n^{-2m/(2m+1)}$ for $h \propto n^{-1/(2m+1)}$.