Stern Gerlach Experiments

Consider now an electron beam, initially prepared by a SG device in the state $\left|\uparrow_x\right\rangle$ which is sent through another SG device with its z-direction aligned such that the eigenvalues $\pm\left(\frac{\hbar}{2}\right)$ of the spin operator S_z, occur with equal frequency, then

$$\varphi(t=0)=\left|\uparrow_x\right\rangle\otimes\left|e\right\rangle=\frac{1}{\sqrt{2}}\left(\left|\uparrow_z\right\rangle+\left|\downarrow_z\right\rangle\right)\otimes\left|e\right\rangle \qquad (0.1)$$

Where $\left|\uparrow_z\right\rangle$, $\left|\downarrow_z\right\rangle$ represent the standard basis $\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}$.

Here also I have included the apparatus (SG device and/or human experimenter plus background environment/rest of the world etc.) to be lumped into one single vector called the environment $\left|e\right\rangle$ which has a very large number of Degrees of freedom – so can easily change state.

The system evolves according to the Schrödinger equation, each electron interacting with the SG analyzer to produce a state for each electron given by:

$$\varphi(t)=\frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle\otimes\left|e_\uparrow\right\rangle+\left|\downarrow\right\rangle\otimes\left|e_\downarrow\right\rangle\right) \qquad (0.2)$$

where $\left|e_\uparrow\right\rangle,\left|e_\downarrow\right\rangle$ are normalised but not necessarily orthogonal so that

$$\left\langle e_\uparrow \mid e_\uparrow\right\rangle=\left\langle e_\downarrow \mid e_\downarrow\right\rangle=1 \qquad (0.3)$$

The density operator for the system is given by definition

$$\rho_{S+E}(t)=\left|\varphi(t)\right\rangle\left\langle\varphi(t)\right| \qquad (0.4)$$

$$\rho_{S+E}(t)=\left(\frac{1}{2}\right)(\left|\uparrow\right\rangle\otimes\left|e_\uparrow\right\rangle+\left|\downarrow\right\rangle\otimes\left|e_\downarrow\right\rangle)(\left\langle\uparrow\right|\otimes\left\langle e_\uparrow\right|+\left\langle\downarrow\right|\otimes\left\langle e_\downarrow\right|) \qquad (0.5)$$

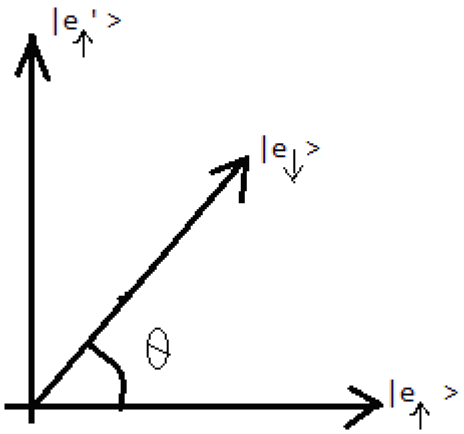
If we restrict our alignment of the SG analyzer to be in either the z or x direction then

$$\rho_{S+E}(t) = \frac{1}{2}(\left|\uparrow\right\rangle\left\langle\uparrow\right|\otimes\left|e_{\uparrow}\right\rangle\left\langle e_{\uparrow}\right| + \left|\uparrow\right\rangle\left\langle\downarrow\right|\otimes\left|e_{\uparrow}\right\rangle\left\langle e_{\downarrow}\right| \qquad (0.6)$$

$$+\left|\downarrow\right\rangle\left\langle\uparrow\right|\otimes\left|e_{\downarrow}\right\rangle\left\langle e_{\uparrow}\right| + \left|\downarrow\right\rangle\left\langle\downarrow\right|\otimes\left|e_{\downarrow}\right\rangle\left\langle e_{\downarrow}\right|) \qquad (0.7)$$

Because we are interested only in what the two state system is doing, and not the environment, one only needs to know the reduced density matrix of the two state system, with the environment states traced out. For this purpose, so that we can form a trace, we need to choose environment basis vectors which are orthogonal. Any normalised orthogonal basis will do since the trace is basis independent.

The diagram below shows one such choice of the basis vectors:



Where

$$\left|e_{\uparrow}\right\rangle \text{ and } \left|e_{\uparrow}'\right\rangle \text{ are orthogonal unit vectors} \qquad (0.8)$$

$$\left\langle e_{\downarrow} \mid e_{\uparrow}'\right\rangle = \cos(90^{0} - \theta) \qquad (0.9)$$

or

$$\left\langle e_{\downarrow} \mid e_{\uparrow}'\right\rangle = \cos(\theta - 90^{0}) \qquad (0.10)$$

$$= \sin\theta \qquad (0.11)$$

And $$\langle e_{\downarrow} | e_{\uparrow} \rangle = \cos\theta \qquad (0.12)$$

$$\langle e_{\uparrow}' | e_{\uparrow} \rangle = 0 \qquad (0.13)$$

The reduced density operator matrix of the two state system is given by

$$\rho_S(t) = Tr_E[\rho_{S+E}(t)] = \langle e_{\uparrow} | \rho_{S+E}(t) | e_{\uparrow} \rangle + \langle e_{\uparrow}' | \rho_{S+E}(t) | e_{\uparrow}' \rangle \qquad (0.14)$$

$$\rho_S(t) = \frac{1}{2}(|\uparrow\rangle\langle\uparrow| \otimes \langle e_{\uparrow} | e_{\uparrow} \rangle \langle e_{\uparrow} | e_{\uparrow} \rangle + |\uparrow\rangle\langle\downarrow| \otimes \langle e_{\uparrow} | e_{\uparrow} \rangle \langle e_{\downarrow} | e_{\uparrow} \rangle$$

$$+ |\downarrow\rangle\langle\uparrow| \otimes \langle e_{\uparrow} | e_{\downarrow} \rangle \langle e_{\uparrow} | e_{\uparrow} \rangle + |\downarrow\rangle\langle\downarrow| \otimes \langle e_{\uparrow} | e_{\downarrow} \rangle \langle e_{\downarrow} | e_{\uparrow} \rangle$$

$$+ |\uparrow\rangle\langle\uparrow| \otimes \langle e_{\uparrow}' | e_{\uparrow} \rangle \langle e_{\uparrow} | e_{\uparrow}' \rangle + |\uparrow\rangle\langle\downarrow| \otimes \langle e_{\uparrow}' | e_{\uparrow} \rangle \langle e_{\downarrow} | e_{\uparrow}' \rangle$$

$$+ |\downarrow\rangle\langle\uparrow| \otimes \langle e_{\uparrow}' | e_{\downarrow} \rangle \langle e_{\uparrow} | e_{\uparrow}' \rangle + |\downarrow\rangle\langle\downarrow| \otimes \langle e_{\uparrow}' | e_{\downarrow} \rangle \langle e_{\downarrow} | e_{\uparrow}' \rangle) \qquad (0.15)$$

Now using the expressions (1.8), (1.17),(1.19) and the fact that outer product terms like $|\uparrow\rangle\langle\downarrow|$ define entries into a matrix like

$$|\uparrow\rangle\langle\downarrow| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

then we have that

$$\rho_s = \frac{1}{2}\begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix}, \qquad (0.16)$$
$$0 < \theta \le \frac{\pi}{2}$$

Now if

$$\langle A \rangle = Tr_s[A\rho_s] \quad (0.17)$$

Where

$$\rho_s = Tr_E[\rho_{S+E}] \quad (0.18)$$

and if we set

$$A = S_z \qquad (0.19)$$

Then

$$\langle S_z \rangle = Tr_s[S_z \rho_s]$$

$$= Tr\left[\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix}\right]$$

$$= Tr_s\left[\frac{\hbar}{4}\begin{pmatrix} 1 & \cos\theta \\ -\cos\theta & -1 \end{pmatrix}\right] = 0 \qquad (0.20)$$

This is the value "expected" since half of the time the electrons will come out as aligned in the positive z-direction and half of the time they will come out aligned in the negative z-direction.

But if we now rotate our SG device into the x-direction and recalculate our expectation value we find that with

$$A = S_x$$

$$\langle S_x \rangle = Tr_s[S_x \rho_s]$$

$$= Tr\left[\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix}\right]$$

$$= Tr_s\left[\frac{\hbar}{4}\begin{pmatrix}\cos\theta & 1\\ 1 & \cos\theta\end{pmatrix}\right]$$

$$= \frac{\hbar}{2}\cos\theta \quad (0.21)$$

Theoretical models[ ] show that decoherence can acts extremely rapidly and hence that in this model decoherence occurs rapidly when

$$\theta \to \frac{\pi}{2}$$

$$\&$$

$$\cos\theta \to 0$$