The corresponding conserved current, calculated from (9.1.16), is

$$\mathcal{J}^{\mu}=\phi_1(\partial^{\mu}\phi_2)-(\partial^{\mu}\phi_1)\phi_2.$$

In this application of Noether's theorem to quantum field theory, $\mathcal{J}^{\mu}$ represents the electromagnetic current associated with the charged particles.

9.2 LORENTZ INVARIANCE AND ANGULAR MOMENTUM

We have just seen that invariances of the action lead to the existence of conserved quantities, and that the conserved quantities derived from invariance under the translations $\bar{x}^{\mu}\to\bar{x}^{\mu}+\varepsilon\delta^{\mu}_{\lambda}$ are the energy and momentum. What are the conserved quantities related to Lorentz invariance?

Lorentz Transformations

It will be helpful for us to explore how finite Lorentz transformations can be built from infinitesimal transformations. We begin with the equation defining a finite Lorentz transformation, $\bar{x}^{\mu}\to x^{\mu}=\Lambda^{\mu}{}_{\nu}\bar{x}^{\nu}$:

$$\Lambda^{\mu}{}_{\alpha}g_{\mu\nu}\Lambda^{\nu}{}_{\beta}=g_{\alpha\beta}. \tag{9.2.1}$$

When written in matrix notation, this is $\Lambda^T\mathbf{g}\Lambda=\mathbf{g}$, or

$$\mathbf{g}\Lambda\mathbf{g}=\Lambda^{-1T}. \tag{9.2.2}$$

Suppose that Λ has the form*

$$\Lambda=e^{\mathbf{A}} \tag{9.2.3}$$

with

$$\mathbf{gAg}=-\mathbf{A}^T. \tag{9.2.4}$$

Then Λ satisfies (9.2.2):

$$\mathbf{g}e^{\mathbf{A}}\mathbf{g}=e^{\mathbf{gAg}}=e^{-\mathbf{A}^T}=(e^{-\mathbf{A}})^T=(e^{\mathbf{A}})^{-1T}. \tag{9.2.5}$$

*The matrix $e^{\mathbf{A}}$ is defined by its power series expansion $e^{\mathbf{A}}=\mathbf{1}+\sum_{n=1}^{\infty}(1/n!)\mathbf{A}^n$. This expansion can be used to prove the equation (9.2.5).

Thus it is plausible that any Lorentz transformation Λ can be represented as an exponential $\exp \mathbf{A}$ with $\mathbf{A}$ satisfying (9.2.4). (Moreover, it is true—as long as Λ is a proper, orthochronous Lorentz transformation.*) This exponential representation is useful because it shows how a finite Lorentz transformation Λ can be formed as the product of many infinitesimal Lorentz transformations

$$\Lambda = e^{\mathbf{A}} = \left[e^{(1/N)\mathbf{A}} \right]^N = \lim_{N\to\infty} \left[\mathbf{1} + \frac{1}{N}\mathbf{A} \right]^N.$$

In order to study Lorentz transformations, we have only to study the "infinitesimal generators" of Lorentz transformations, that is, the matrices $\mathbf{A}$ that satisfy the simple linear equation (9.2.4). Let us write (9.2.4) as $\mathbf{Ag} = -\mathbf{gA}^T$:

$$A^{\mu}{}_{\alpha} g^{\alpha\nu} = -g^{\mu\alpha} A^{\nu}{}_{\alpha},$$

or

$$A^{\mu\nu} = -A^{\nu\mu}. \tag{9.2.6}$$

There are six linearly independent solutions to (9.2.6):

$$A^{\mu\nu} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ldots.$$

We name these six matrices $\mathbf{M}_{01}$, $\mathbf{M}_{02}$, $\mathbf{M}_{03}$, $\mathbf{M}_{12}$, $\mathbf{M}_{13}$, $\mathbf{M}_{23}$. For notational convenience, we also define $\mathbf{M}_{\alpha\beta} = 0$ for $\alpha = \beta$, $\mathbf{M}_{\alpha\beta} = -\mathbf{M}_{\beta\alpha}$ for $\alpha > \beta$. By definition, then, the matrix elements $(M_{\alpha\beta})^{\mu}{}_{\nu}$ of the infinitesimal generator $\mathbf{M}_{\alpha\beta}$ are

$$(M_{\alpha\beta})^{\mu}{}_{\nu} = -g^{\mu}_{\alpha} g_{\beta\nu} + g^{\mu}_{\beta} g_{\alpha\nu}. \tag{9.2.7}$$

*The representation (9.2.3), (9.2.4) can be easily established, as long as the matrix elements of $\Lambda - 1$ are small enough, by defining $\mathbf{A} = \ln \Lambda = -(1-\Lambda) - \frac{1}{2}(1-\Lambda)^2 - \frac{1}{3}(1-\Lambda)^3 - \cdots$. For our purposes this is all that is necessary. However, one can extend the representation to all proper orthochronous Lorentz transformations by an explicit construction. For this purpose it is best to use the correspondence between the proper orthochronous Lorentz group and the group of 2×2 complex matrices with determinant 1, $SL(2,C)$. [See, for example, R. F. Streater and A. S. Wightman, *PCT, Spin and Statitics, and All That* (Benjamin, New York, 1964).]

Transformations of Vectors and Tensors

We have seen that a finite Lorentz transformation has the form $e^{\mathbf{A}}$ where $\mathbf{A}$ is a linear combination of the six matrices $\mathbf{M}_{\alpha\beta}$:

$$\Lambda = e^{1/2\,\omega^{\alpha\beta}\mathbf{M}_{\alpha\beta}},$$

$$\omega_{\alpha\beta} = -\omega_{\beta\alpha}. \tag{9.2.8}$$

One can specify the Lorentz transformation by giving the six independent parameters $\omega^{01}, \omega^{02}, \omega^{03}, \omega^{12}, \omega^{13}, \omega^{23}$.

A vector V^{μ} transforms under the Lorentz transformation specified by $\omega^{\alpha\beta}$ according to $\bar{V}^{\mu} \to V^{\mu}$ with

$$V^{\mu} = \left[e^{1/2\,\omega^{\alpha\beta}\mathbf{M}^{\alpha\beta}} \right]^{\mu}{}_{\nu} \bar{V}^{\nu}. \tag{9.2.9}$$

The transformation law for a second rank tensor $T^{\mu\nu}$ can be written in the same way if we think of the pair of indices (μ, ν) as one index that takes 16 values. We simply need new 16×16 generating matrices.

$$(\tilde{\mathbf{M}}_{\alpha\beta})^{\mu\nu}{}_{\rho\sigma} = (\mathbf{M}_{\alpha\beta})^{\mu}{}_{\rho} g^{\nu}_{\sigma} + g^{\mu}_{\rho} (\mathbf{M}_{\alpha\beta})^{\nu}{}_{\sigma}. \tag{9.2.10}$$

Then

$$T^{\mu\nu} = \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} \bar{T}^{\rho\sigma} = \left[e^{1/2\,\omega^{\alpha\beta}\tilde{\mathbf{M}}_{\alpha\beta}} \right]^{\mu\nu}{}_{\rho\sigma} \bar{T}^{\rho\sigma}. \tag{9.2.11}$$

To verify this formula, note that $\mathbf{M}$ has the form $\mathbf{C}+\mathbf{D}$, and that $\exp(\mathbf{C}+\mathbf{D}) = \exp(\mathbf{C})\exp(\mathbf{D})$, since $\mathbf{CD} = \mathbf{DC}$.

It should be apparent that the transformation law for a tensor of any rank can be written in a form like (9.2.11). In the case of a scalar field $T(x)$, the corresponding generating matrices $\tilde{\mathbf{M}}_{\alpha\beta}$ are 1×1 matrices (i.e., numbers) whose value is zero.

There is another class of fields other than tensors that is used in relativistic quantum field theory. These fields are called spinors, and also transform according to a law of the form

$$\bar{\psi}_K(\bar{x}) \to \psi_K(x) = \left[e^{1/2\,\omega^{\alpha\beta}\tilde{\mathbf{M}}_{\alpha\beta}} \right]_{KL} \psi_L(x),$$

but with another form for the generating matrices $\tilde{\mathbf{M}}_{\alpha\beta}$.

Lorentz Invariance in a General Theory

Suppose that we are dealing with a relativistic field theory involving several fields, $\phi(x)$, $T^{\nu\mu}(x)$, $A^{\mu}(x)$, etc. In order to have a flexible notation, we give each component of each field a new name, $\phi_J(x)$, $J=1,2,\ldots,N$. Then the transformation law for the fields under Lorentz transformations

$$\bar{x}^{\mu}\to x^{\mu}=\left[e^{1/2\omega^{\alpha\beta}\mathbf{M}_{\alpha\beta}}\right]^{\mu}{}_{\nu}\bar{x}^{\nu} \tag{9.2.12}$$

has the form

$$\bar{\phi}_J(\bar{x})\to\phi_J(x)=\left[e^{1/2\omega^{\alpha\beta}\tilde{\mathbf{M}}_{\alpha\beta}}\right]_{JK}\bar{\phi}_K(\bar{x}). \tag{9.2.13}$$

The generating matrices $\tilde{\mathbf{M}}_{\alpha\beta}$ have a block diagonal form, so that $(\tilde{M}_{\alpha\beta})_{JK}=0$ unless $\phi_J(x)$ and $\phi_K(x)$ are two components of the same vector or tensor field.

Application of Noether's Theorem

If the Lagrangian is a scalar under Lorentz transformations, then Noether's theorem tells us that there are six conserved quantities $J_{\alpha\beta}$, one for each independent parameter $\omega^{\alpha\beta}$. (We take $J_{\alpha\beta}=-J_{\beta\alpha}$ for convenience.) The corresponding conserved currents $J_{\alpha\beta}{}^{\mu}$ can be calculated by using (9.1.16) and taking ε equal to one of the parameters $\omega_{\alpha\beta}=-\omega_{\beta\alpha}$, with the other parameters equal to zero.

$$\frac{\partial}{\partial\varepsilon}x^{\mu}=(M_{\alpha\beta})^{\mu}{}_{\nu}x^{\nu},$$

$$\frac{\partial}{\partial\varepsilon}\phi_J=(\tilde{M}_{\alpha\beta})_{JK}\phi_K.$$

Thus

$$\begin{aligned}
J_{\alpha\beta}{}^{\mu}&=\mathfrak{L}(M_{\alpha\beta})^{\mu}{}_{\nu}x^{\nu}\\
&\quad+\frac{\partial\mathfrak{L}}{\partial(\partial_\mu\phi_J)}\left[(\tilde{M}_{\alpha\beta})_{JK}\phi_K-(\partial_\nu\phi_J)(M_{\alpha\beta})^{\nu}{}_{\lambda}x^{\lambda}\right]\\
&=x_\alpha\left[g^{\mu}_{\beta}\mathfrak{L}-(\partial_\beta\phi_J)\frac{\partial\mathfrak{L}}{\partial(\partial_\mu\phi_J)}\right]\\
&\quad-x_\beta\left[g^{\mu}_{\alpha}\mathfrak{L}-(\partial_\alpha\phi_J)\frac{\partial\mathfrak{L}}{\partial(\partial_\mu\phi_J)}\right]\\
&\quad+\frac{\partial\mathfrak{L}}{\partial(\partial_\mu\phi_J)}(\tilde{M}_{\alpha\beta})_{JK}\phi_K
\end{aligned}$$

or

$$J_{\alpha\beta}{}^{\mu} = x_\alpha T_\beta{}^{\mu} - x_\beta T_\alpha{}^{\mu} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_J)} (\tilde{M}_{\alpha\beta})_{JK} \phi_K. \tag{9.2.14}$$

The current $J_{\alpha\beta}{}^{\mu}$ is called the angular momentum current, and the conserved tensor $J_{\alpha\beta} = \int d^3x J_{\alpha\beta}{}^0$ is called the angular momentum. We relate $J_{\alpha\beta}$ to the angular momentum $\mathbf{x} \times \mathbf{p}$ of particle mechanics in the next section.

9.3 PHYSICAL INTERPRETATION OF THE ANGULAR MOMENTUM TENSOR

Theories with Scalar Fields

In order to get a feeling for what the conserved quantities $J_{\alpha\beta}$ are, let us look first at theories with only scalar fields. For example, we can think of the theory of elastic materials, in which we deal with three scalar fields $R_a(x)$. Since all of the fields are scalars, the matrix $\tilde{\mathbf{M}}_{\alpha\beta}$ in (9.2.14) is zero and

$$J_{\alpha\beta}{}^{\mu} = x_\alpha T_\beta{}^{\mu} - x_\beta T_\alpha{}^{\mu}. \tag{9.3.1}$$

What is the quantity J_{12}? It is conserved because of the invariance of the action under Lorentz transformations

$$\Lambda^\mu{}_\nu = \left[e^{\omega \mathbf{M}_{12}} \right]^\mu{}_\nu. \tag{9.3.2}$$

If we use the definition (9.2.7) of $\mathbf{M}_{12}$ and expand the exponential in its power series, we find

$$\Lambda^\mu{}_\nu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega & 0 \\ 0 & \sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{9.3.3}$$

Thus conservation of J_{12} arises from invariance of the action under rotations about the x^3-axis. Looking at the expression (9.3.1) for the density $J_{12}{}^0$, we find that it is related to the momentum density T^{k0} by

$$J_{12}{}^0 = x^1 T^{20} - x^2 T^{10}. \tag{9.3.4}$$

This is just the third component of $\mathbf{x}\times\mathbf{p}$, where $p^k=T^{k0}$ is the momentum density.

These results generalize nicely. If we define a three-vector $J_i=\frac{1}{2}\varepsilon_{ijk}J_{jk}$, then

$$\mathbf{J}=\int d\mathbf{x}\,\mathbf{x}\times\mathbf{p} \tag{9.3.5}$$

and conservation of the component $\mathbf{n}\cdot\mathbf{J}$ arises from invariance of the action under rotations about the vector $\mathbf{n}$. If we are describing an elastic material, then each small piece of material contributes to the angular momentum an amount $\mathbf{x}\times\mathbf{p}$ equal to its "orbital" angular momentum. There is no allowance in (9.3.5) for any extra contributions due to spinning motions of the small piece or the spin angular momentum of the electrons or nuclei in the material.

What about the other components of $J_{\alpha\beta}$: J_{01}, J_{02}, J_{03}? The conservation of these components is due to the invariance of the action under pure Lorentz boosts like

$$\Lambda^{\mu}{}_{\nu}=\left[e^{\omega M_{01}}\right]^{\mu}{}_{\nu}=\begin{pmatrix}\cosh\omega & -\sinh\omega & 0 & 0\\ -\sinh\omega & \cosh\omega & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. \tag{9.3.6}$$

Let us write out J_{0k} as given by (9.3.1):

$$J_{0k}=\int d\mathbf{x}(-tT^{k0}+x^kT^{00}). \tag{9.3.7}$$

The quantity $\int d\mathbf{x}\,T^{k0}$ is the total momentum P^k of the system. The quantity $\int d\mathbf{x}\,x^kT^{00}$ is the total energy E of the system times the position $X^k(t)$ of the center of energy. Thus

$$J_{0k}=-tP^k+EX^k(t). \tag{9.3.8}$$

Conservation of J_{0k} means that

$$0=\frac{d}{dt}J_{0k}=-P^k+E\frac{dX^k}{dt}. \tag{9.3.9}$$

Thus the velocity $d\mathbf{X}/dt$ of the center of energy of the system is a constant equal to $\mathbf{P}/E$, just as it is for point particles.

Before we leave the scalar field case, we should recall that the momentum tensor for elastic materials was symmetric. This was no accident. It is a consequence of angular momentum conservation, since

$$\begin{aligned}0 = \partial_\mu J_{\alpha\beta}{}^\mu &= \partial_\mu\left[x_\alpha T_\beta{}^\mu - x_\beta T_\alpha{}^\mu\right]\\ &= T_{\beta\alpha} - T_{\alpha\beta}.\end{aligned} \tag{9.3.10}$$

When we deal with theories with vector fields, we will find that the expression (9.3.1) for $J_{\alpha\beta}{}^\mu$ is modified, so that (9.3.10) no longer holds.

Theories with Vector Fields

The physics of angular momentum becomes more interesting in theories with vector, tensor, or spinor fields. In such theories the angular momentum current has the form (9.2.14),

$$J_{\alpha\beta}{}^\mu = x_\alpha T_\beta{}^\mu - x_\beta T_\alpha{}^\mu + S_{\alpha\beta}{}^\mu, \tag{9.3.11}$$

where

$$S_{\alpha\beta}{}^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_J)}\left[\tilde{M}_{\alpha\beta}\right]_{JK}\phi_K. \tag{9.3.12}$$

We have seen that the terms $x_\alpha T_\beta{}^\mu - x_\beta T_\alpha{}^\mu$ can be interpreted as the orbital angular momentum associated with the momentum current $T^{\mu\nu}$. Thus the remaining term is $S_{\alpha\beta}{}^\mu$ is best described as an "intrinsic" or "spin" angular momentum carried by the tensor fields. The terminology here is meant to suggest the situation in quantum mechanics, where an electron has an orbital angular momentum $\mathbf{x} \times \mathbf{p}$ plus a spin angular momentum $\mathbf{S}$.*

Electrodynamics

The most important classical field theory in which the fields carry spin angular momentum is electrodynamics. We recall the Lagrangian (8.3.4) for the electromagnetic field coupled to charged matter,

$$\mathcal{L} = -\rho U(G_{ab}, R_a) - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{J}_\mu A^\mu. \tag{9.3.13}$$

The spin current $S_{\alpha\beta}{}^\mu$ can be calculated using (9.3.12) and a knowledge of

*This terminology is actually more than just suggestive. In the description of electrons in quantum field theory the operator that measures the electron spin is precisely $\int d\mathbf{x}\, S_{\alpha\beta}{}^0$.

the transformation law for the fields,

$$\begin{pmatrix} \bar{R}_a \\ \bar{A}^\mu \end{pmatrix} \to \begin{pmatrix} R_a \\ A^\mu \end{pmatrix} = e^{1/2\omega^{\alpha\beta}\tilde{\mathbf{M}}_{\alpha\beta}} \begin{pmatrix} \bar{R}_a \\ \bar{A}^\mu \end{pmatrix},$$

$$\tilde{M}_{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & [\mathbf{M}_{\alpha\beta}]^\mu{}_\nu \end{pmatrix}.$$

We find

$$\begin{aligned} S_{\alpha\beta}{}^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\lambda)} [M_{\alpha\beta}]^\lambda{}_\sigma A^\sigma \\ &= -F^\mu{}_\lambda\left[-g^\lambda_\alpha g_{\beta\sigma} + g^\lambda_\beta g_{\alpha\sigma}\right] A^\sigma, \end{aligned}$$

or

$$S_{\alpha\beta}{}^\mu = F^\mu{}_\alpha A_\beta - F^\mu{}_\beta A_\alpha. \tag{9.3.14}$$

In order to interpret $S_{\alpha\beta}{}^\mu$, let us calculate the spin density

$$s_j = \tfrac{1}{2}\varepsilon_{jkl} S_{kl}{}^0 \tag{9.3.15}$$

for an electromagnetic wave. Using (9.3.14) we find

$$\mathbf{s} = \mathbf{E} \times \mathbf{A}. \tag{9.3.16}$$

To describe a circularly polarized plane wave traveling in the z-direction, we can choose* a potential

$$A^\mu(x) = (0, a\cos[\omega(z-t)], \mp a\sin[\omega(z-t)], 0). \tag{9.3.17}$$

The minus (plus) sign corresponds to a left (right) circularly polarized wave. The corresponding electric field is

$$E_k = \partial^0 A^k - \partial^k A^0 = -\partial_0 A^k. \tag{9.3.18}$$

Thus the spin density carried by this wave is

$$\mathbf{s} = \pm a^2 \omega \hat{\mathbf{z}}, \tag{9.3.19}$$

where $\hat{\mathbf{z}}$ is a unit vector pointing in the z-direction.

*The spin density s is gauge dependent, so we have made a specific choice of gauge. However, s is unaffected by a gauge change of the form $A^\mu(x) \to A^\mu(x) + \varepsilon\partial^\mu \cos[\omega(z-t)+\phi]$.

This result can be understood very simply by thinking of the wave as a beam of photons. Each photon carries an energy $\hbar\omega$. If there are N photons per unit volume, the energy density is $T^{00} = N\hbar\omega$. On the other hand, the energy density for this wave, calculated using (8.3.7), is $T^{00} = a^2\omega^2$. Thus $a^2 = N\hbar/\omega$. This result enables us to write the spin density in terms of the photon density N:

$$\mathbf{s} = \pm N\hbar\hat{\mathbf{z}}. \tag{9.3.20}$$

Apparently each left circularly polarized photon traveling in the z-direction carries spin angular momentum $+\hbar\hat{\mathbf{z}}$; each right polarized photon carries angular momentum $-\hbar\hat{\mathbf{z}}$.

We have given a simple picture of the spin density of the electromagnetic field by using the language of quantum electrodynamics (spinning photons). Nevertheless, this spin density is a feature of classical electrodynamics. Furthermore, the angular momentum of a circularly polarized light beam can be measured in a classical macroscopic experiment. One passes the beam through a "half wave plate," which changes left polarized light into right polarized light. In the process, the plate absorbs $2\hbar$ of angular momentum from each photon that passes through it. If the plate is suspended on a thin thread, it can be observed to rotate.*

9.4 THE SYMMETRIZED MOMENTUM TENSOR

We have seen that the angular momentum tensor in theories with vector (or tensor or spinor) fields contains a spin term $S_{\alpha\beta}{}^{\mu}$. In such theories the tensor $T^{\mu\nu}$, which we will call the "canonical" momentum tensor, is not symmetric.

$$0 = \partial_\mu J_{\alpha\beta}{}^{\mu} = \partial_\mu\left[x_\alpha T_\beta{}^{\mu} - x_\beta T_\alpha{}^{\mu} + S_{\alpha\beta}{}^{\mu}\right],$$

so that

$$T_{\alpha\beta} - T_{\beta\alpha} = \partial_\mu S_{\alpha\beta}{}^{\mu}.$$

In this section we find that it is always possible to define a new, symmetric momentum tensor $\Theta^{\mu\nu}$ which leads to the same total momentum $\int d\mathbf{x}\,\Theta^{\mu 0}$ as the canonical tensor.†

*R. A. Beth, *Phys. Rev.* **50**, 115 (1936); A. H. S. Holbourn, *Nature* **137**, 31 (1936).

†The construction of the symmetrized momentum tensor is due to F. J. Belinfante, *Physica* **7**, 449 (1940), and L. Rosenfeld, *Mem. Acad. R. Belg. Sci.* **18**, No. 6 (1940).

Why is $\Theta^{\mu\nu}$ important? It would seem that there is little basis for preferring one of the two tensors $\Theta^{\mu\nu}$ and $T^{\mu\nu}$ if they lead to the same total momentum and energy. The only difference lies in the different description of where the energy is located. But when we investigate gravity in Chapter 11, the location of energy will be important, since energy is the source of the gravitational field. At this point it is sufficient to say that in the theory of general relativity, the tensor $\Theta^{\mu\nu}$ rather than $T^{\mu\nu}$ is the source of the gravitational field. In electrodynamics the symmetric tensor $\Theta^{\mu\nu}(x)$ has the additional advantage that it is gauge invariant while $T^{\mu\nu}$ is not.

The construction of $\Theta^{\mu\nu}$ begins with the spin tensor $S_{\alpha\beta}{}^{\lambda}$. From its definition (9.3.12) we see that $S_{\alpha\beta}{}^{\lambda}$ is antisymmetric in its first two indices. Thus the tensor

$$G_{\mu\nu\rho}=\tfrac{1}{2}\left[S_{\nu\rho\mu}+S_{\mu\rho\nu}-S_{\mu\nu\rho}\right] \tag{9.4.1}$$

is antisymmetric in its last two indices. We define

$$\Theta^{\mu\nu}=T^{\mu\nu}+\partial_{\rho}G^{\mu\nu\rho}. \tag{9.4.2}$$

Since $G^{\mu\nu\rho}$ is antisymmetric in the indices $\nu\rho$, the conservation of $\theta^{\mu\nu}$ is equivalent to conservation of $T^{\mu\nu}$:

$$\partial_{\nu}\Theta^{\mu\nu}=\partial_{\nu}T^{\mu\nu}+\partial_{\nu}\partial_{\rho}G^{\mu\nu\rho}=\partial_{\nu}T^{\mu\nu}=0. \tag{9.4.3}$$

Using the antisymmetry of $G^{\mu\nu\rho}$ and an integration by parts,* we find that $\Theta^{\mu\nu}$ and $T^{\mu\nu}$ give the same total momentum:

$$\int d\mathbf{x}\,\Theta^{\mu 0}-\int d\mathbf{x}\,T^{\mu 0}=\int d\mathbf{x}\,\partial_{\rho}G^{\mu 0\rho}=\sum_{k=1}^{3}\int d\mathbf{x}\,\partial_{k}G^{\mu 0k}=0. \tag{9.4.4}$$

Finally, we can use conservation of momentum and angular momentum to show that $\Theta^{\mu\nu}$ is symmetric:

$$\Theta^{\mu\nu}-\Theta^{\nu\mu}=T^{\mu\nu}-T^{\nu\mu}-\partial_{\rho}S^{\mu\nu\rho},$$

but

$$\begin{aligned}0&=\partial_{\rho}J^{\nu\mu\rho}\\&=\partial_{\rho}\left[x^{\nu}T^{\mu\rho}-x^{\mu}T^{\nu\rho}+S^{\nu\mu\rho}\right]\\&=T^{\mu\nu}-T^{\nu\mu}-\partial_{\rho}S^{\mu\nu\rho},\end{aligned}$$

*We assume that the fields fall off sufficiently rapidly as $|\mathbf{x}|\to\infty$ so that the surface term in (9.4.4) vanishes.

so that

$$\Theta^{\mu\nu}-\Theta^{\nu\mu}=0. \tag{9.4.5}$$

Angular Momentum

In theories with scalar fields only, the canonical momentum tensor $T^{\mu\nu}$ is symmetric and the "canonical" angular momentum tensor $J^{\alpha\beta\lambda}$ is equal to $x^\alpha T^{\beta\lambda}-x^\beta T^{\alpha\lambda}$. Now that we have a symmetric momentum tensor $\theta^{\mu\nu}$ at hand for any theory, we are tempted to use it to define an angular momentum tensor

$$J_\theta^{\alpha\beta\lambda}=x^\alpha\theta^{\beta\lambda}-x^\beta\theta^{\alpha\lambda}. \tag{9.4.6}$$

This tensor is distinguished by the lack of a spin term like that occurring in the canonical angular momentum tensor

$$J^{\alpha\beta\lambda}=x^\alpha T^{\beta\lambda}-x^\beta T^{\alpha\lambda}+S^{\alpha\beta\lambda}.$$

Thus the angular momentum $J_\theta{}^{\alpha\beta}=\int d\mathbf{x} J_\theta^{\alpha\beta 0}$ contains only "orbital" angular momentum.

Can we get away with this? We first note that $J_\theta^{\alpha\beta\lambda}$ is conserved as a direct consequence of the fact that $\theta^{\mu\nu}$ is symmetric and conserved. In order to compare the total angular momenta found by integrating $J_\theta^{\alpha\beta\lambda}$ and $J^{\alpha\beta\lambda}$, we can insert the definition (9.4.2) of $\theta^{\mu\nu}$ into (9.4.6):

$$\begin{aligned}J_\theta^{\alpha\beta\lambda}&=x^\alpha T^{\beta\lambda}-x^\beta T^{\alpha\lambda}+x^\alpha\partial_\rho G^{\beta\lambda\rho}-x^\beta\partial_\rho G^{\alpha\lambda\rho}\\&=x^\alpha T^{\beta\lambda}-x^\beta T^{\alpha\lambda}-G^{\beta\lambda\alpha}+G^{\alpha\lambda\beta}\\&\quad+\partial_\rho(x^\alpha G^{\beta\lambda\rho}-x^\beta G^{\alpha\lambda\rho}).\end{aligned}$$

From the definition (9.4.1) of $G_{\mu\nu\rho}$ we can deduce that $G^{\alpha\lambda\beta}-G^{\beta\lambda\alpha}=S^{\alpha\beta\lambda}$, so that

$$J_\theta^{\alpha\beta\lambda}=J^{\alpha\beta\lambda}+\partial_\rho(x^\alpha G^{\beta\lambda\rho}-x^\beta G^{\alpha\lambda\rho}). \tag{9.4.7}$$

If we now recall that $G^{\beta\lambda\rho}$ is antisymmetric in its last two indices and use an integration by parts we find

$$\begin{aligned}&\int d\mathbf{x} J_\theta{}^{\alpha\beta 0}-\int d\mathbf{x} J^{\alpha\beta 0}\\&=\sum_{\rho=1}^{3}\int d^3x\,\partial_\rho(x^\alpha G^{\beta 0\rho}-x^\beta G^{\alpha 0\rho})\\&=0.\end{aligned} \tag{9.4.8}$$

Thus we can indeed use $J_{\theta}^{\alpha\beta\lambda}$ in place of $J^{\alpha\beta\lambda}$ as the angular momentum current, if we so wish.

A Mechanical Analogy

A simple mechanical example may serve to clarify the formal construction just presented. Imagine that on a Sunday morning everybody in London drives his car continuously around his block in a counterclockwise direction, as shown in Figure 9.1. The cars circling block number (N_1,N_2) have a net momentum of zero and a net angular momentum $s(N_1,N_2)$, where $s(\mathbf{N})$ is proportional to the number of cars circling the block.

If $s(\mathbf{N})$ is a slowly varying function of $\mathbf{N}$, it is sensible to give a macroscopic description of this situation. We may say that the macroscopic momentum density T^{k0} is zero and that the macroscopic angular momentum density is $J^{120} = S^{120} = s(\mathbf{N})$.

On the other hand, we can base a macroscopic description on a street by street accounting instead of a block by block accounting. Suppose, as shown in Figure 9.1, that $s(\mathbf{N})$ increases as one moves east. Then each north-south street carries a net flow of cars toward the south. The magnitude of this flow is proportional to $\partial s/\partial N_1$. Thus there is "really" an

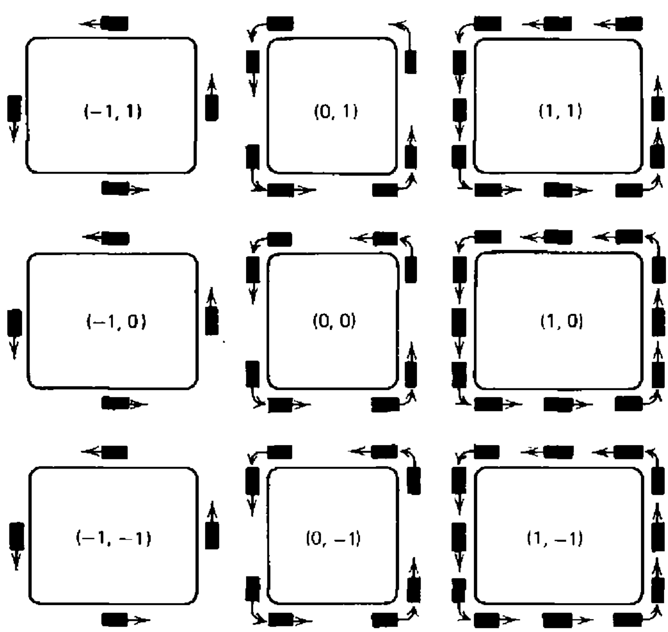

Figure 9.1 Angular momentum of traffic flow in London.

average momentum per block of

$$\theta^{10}=(\text{const})\frac{\partial s}{\partial N_2}, \quad \theta^{20}=-(\text{const})\frac{\partial s}{\partial N_1}.$$

A straightforward calculation shows that the constant appearing here is $\frac{1}{2}$. This result is the same as the formal definition (9.4.2) of θ^{j0} if we set $S^{0k\mu}=0$:

$$\theta^{j0}=T^{j0}+\tfrac{1}{2}\partial_k S^{jk0}. \tag{9.4.10}$$

When the traffic pattern is viewed in this street by street manner, the angular momentum of the traffic flow arises from the net flow of cars counterclockwise around high s areas and clockwise around low s areas:

$$J_\theta^{120}=N^1\Theta^{20}-N^2\Theta^{10}.$$

The reader may ponder whether the $(T^{\mu\nu},J^{\mu\nu\lambda})$ description or the $(\Theta^{\mu\nu},J_\theta^{\mu\nu\lambda})$ description is preferable in this situation.

9.5 THE SYMMETRIZED MOMENTUM TENSOR IN ELECTRODYNAMICS

Let us try out the formalism for constructing a symmetric momentum tensor by using it on electrodynamics. We recall from Section 9.3 that the Lagrangian

$$\mathcal{L}=-\rho U(G_{ab},R_a)-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\mathcal{J}_\mu A^\mu \tag{9.5.1}$$

for the electromagnetic field interacting with charged matter leads to a spin tensor

$$S^{\alpha\beta\mu}=F^{\mu\alpha}A^\beta-F^{\mu\beta}A^\alpha. \tag{9.5.2}$$

Thus the tensor $G^{\mu\nu\rho}$ defined by (9.4.1) is

$$G^{\mu\nu\rho}=-F^{\nu\rho}A^\mu. \tag{9.5.3}$$

The canonical momentum tensor is given by (8.3.6):

$$T^{\mu\nu}=T_M^{\mu\nu}+(\partial^\mu A_\rho)F^{\nu\rho}-\tfrac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}+A^\mu\mathcal{J}^\nu, \tag{9.5.4}$$

where

$$T_M^{\mu\nu} = \rho U u^\mu u^\nu + 2\rho(\partial^\mu R_a)(\partial^\nu R_b)\frac{\partial U}{\partial G_{ab}}$$

is the momentum tensor of the matter. We form $\theta^{\mu\nu}$ by adding $\partial_\rho G^{\mu\nu\rho}$ to $T^{\mu\nu}$:

$$\begin{aligned}\Theta^{\mu\nu} = T_M^{\mu\nu} &+ (\partial^\mu A_\rho - \partial_\rho A^\mu)F^{\nu\rho} - \tfrac{1}{4} g^{\mu\nu} F_{\alpha\beta}F^{\alpha\beta} \\ &+ A^\mu(\mathcal{J}^\nu - \partial_\rho F^{\nu\rho}).\end{aligned}$$

In the second term we recognize $F^{\mu\rho}$, and we note that the last term is zero on account of the equation of motion* for $F^{\nu\rho}$. Thus

$$\begin{aligned}\Theta^{\mu\nu} &= \Theta_M^{\mu\nu} + \Theta_E^{\mu\nu}, \\ \Theta_M^{\mu\nu} &= T_M^{\mu\nu}, \\ \Theta_E^{\mu\nu} &= F^\mu{}_\rho F^{\nu\rho} - \tfrac{1}{4} g^{\mu\nu} F_{\alpha\beta}F^{\alpha\beta}.\end{aligned} \tag{9.5.5}$$

Three nice things have happened here. The new tensor is symmetric, as expected. It is gauge invariant. Finally, it splits into two terms, one involving only the matter fields R_a, the other involving only the electromagnetic field $F^{\mu\nu}$. Thus one may speak of the energy of matter Θ_M^{00} and the energy of the electromagnetic field Θ_E^{00} without needing an interaction energy like $T_I^{00} = A^0\mathcal{J}^0$.

The electromagnetic part of $\Theta^{\mu\nu}$ is probably already familiar to the reader. The energy density is

$$\Theta_E^{00} = \tfrac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2). \tag{9.5.6}$$

The energy current, usually called Poynting's vector, is

$$\Theta_E^{0k} = (\mathbf{E} \times \mathbf{B})_k. \tag{9.5.7}$$

The momentum density Θ_E^{k0} is also equal to Poynting's vector, since $\Theta^{\mu\nu}$ is symmetric. Finally, the momentum current, often called the Maxwell stress

*When the Lagrangian is a function of the fields and their first derivatives, so is the canonical momentum tensor $T^{\mu\nu}$. However, the symmetrized tensor $\Theta^{\mu\nu}$ can depend also on the second derivatives of the fields, since its definition involves derivatives of the spin current $S^{\alpha\beta\mu}$. Fortunately, the second derivatives of the fields can usually be eliminated from $\Theta^{\mu\nu}$ by using the equations of motion, as was done here.

tensor, is

$$\Theta^{ij} = -E_iE_j - B_iB_j + \tfrac{1}{2}\delta_{ij}(\mathbf{E}^2 + \mathbf{B}^2). \tag{9.5.8}$$

We may recall from (8.3.9) that the rate of momentum transfer from the electromagnetic field to matter is

$$\partial_\nu \Theta_M^{\mu\nu} = F^{\mu\nu} \mathcal{J}_\nu. \tag{9.5.9}$$

(This is just the force law usually thought of as defining $F^{\mu\nu}(x)$.) Since $\partial_\nu(\Theta_M^{\mu\nu} + \Theta_E^{\mu\nu}) = 0$, we can also write

$$-\partial_\nu \Theta_E^{\mu\nu} = F^{\mu\nu} \mathcal{J}_\nu. \tag{9.5.10}$$

It is left as an exercise (Problem 1) to verify (9.5.10) directly from the equations of motion for $F^{\mu\nu}$.

Angular Momentum

If one uses $\Theta_E^{\mu\nu}$ to describe the momentum flow in an electromagnetic field, then one might naturally use $J_\theta^{\alpha\beta\mu} = x^\alpha\theta^{\beta\mu} - x^\beta\theta^{\alpha\mu}$ to describe the flow of angular momentum. Consider the case of a left circularly polarized plane wave propagating in the z-direction, as discussed in Section 9.3. We recall that such a wave carries an angular momentum density

$$J^{120} = S^{120} = +N\hbar,$$

where $N = T^{00}/\hbar\omega$ is the number of photons per unit volume. However, direct calculation of J_θ gives

$$J_\Theta^{120} = x^1\Theta^{20} - x^2\Theta^{10} = 0$$

since the momentum density $\Theta^{j0} = (\mathbf{E}\times\mathbf{B})_j$ is a vector pointing precisely in the z-direction.

What happened to the spin angular momentum of the circularly polarized photons? The example in Section 9.4 of the traffic flow in London should suggest the answer. Near the edge of a beam of light the fields $F^{\mu\nu}$ must behave in a complicated way in order to satisfy Maxwell's equations while changing from a plane wave inside the beam to zero outside the beam. Thus the energy current $\mathbf{E}\times\mathbf{B}$ near the edge need not be precisely in the z-direction. In fact, there must be an energy flow in a counterclockwise direction around the edge of the beam which gives the angular momentum of the beam. Thus by changing descriptions of the momentum current, we have assigned the angular momentum to the edge of the polarized beam of light instead of to the middle.

9.6 A SYMMETRIZED MOMENTUM TENSOR FOR NONRELATIVISTIC SYSTEMS

We have seen that every field theory derived from an action which is invariant under translations and Lorentz transformations possesses a conserved symmetric momentum tensor $\theta^{\mu\nu}$. If the action is invariant under translations and rotations but not Lorentz boosts, a similar but less powerful theorem can be proved.

Translational invariance tells us that there is a conserved momentum current $T^{k\mu}(k=1,2,3;\mu=0,1,2,3)$. If vector fields are involved in the theory, the stress tensor T^{kl} will not be symmetric. However, rotational invariance implies the existence of a conserved angular momentum current $J^{kl\mu}$, which we choose to write in the form

$$J^{kl\mu}=x^kT^{l\mu}-x^lT^{k\mu}+S^{kl\mu}. \tag{9.6.1}$$

With the aid of the spin current $S^{kl\mu}$ we can construct a new momentum current $\theta^{k\mu}$ which gives the same total momentum as $T^{k\mu}$ and for which the stress θ^{kl} is symmetric. To do so, we define*

$$\theta^{k0}=T^{k0}+\tfrac{1}{2}\,\partial_l S^{kl0}, \tag{9.6.2}$$

$$\theta^{kl}=T^{kl}-\tfrac{1}{2}\,\partial_0 S^{kl0}+\tfrac{1}{2}\,\partial_j\left[S^{ljk}+S^{kjl}-S^{klj}\right]. \tag{9.6.3}$$

Note that θ^{k0} is precisely the expression (9.4.10) we obtained for the "real" momentum density in the example of traffic flow in London.

We must show that $\theta^{k\mu}$ has the desired properties. First note that $\theta^{k\mu}$ is conserved, because $T^{k\mu}$ is conserved and $S^{kj\mu}$ is antisymmetric in its first two indices:

$$\partial_\mu\theta^{k\mu}=\partial_\mu T^{k\mu}+\tfrac{1}{2}\,\partial_l\partial_j S^{ljk}=0.$$

Second, the total momenta defined by θ^{k0} and T^{k0} are the same because $(\theta^{k0}-T^{k0})$ is a divergence. Finally, we prove that $\theta^{kl}=\theta^{lk}$ by exploiting the conservation of momentum and angular momentum:

$$0=\partial_\mu J^{lk\mu}=T^{kl}-T^{lk}-\partial_\mu S^{kl\mu}=\theta^{kl}-\theta^{lk}.$$

*This construction is not useful in a Lorentz invariant theory because $\theta^{k\mu}$ is not part of a tensor under Lorentz transformations. Also, nothing is said about the relation of the momentum density θ^{k0} to an energy current θ^{0k}. The present definition of $\theta^{k\mu}$ can be obtained from the Lorentz covariant definition by setting $S^{0k\mu}=0$.