FINAL PROJECT

Let us consider how to solve the following one-dimensional PDE numerically using the spectral method.

$$\begin{cases} -u_{xx} = \pi^2 \sin(\pi x), x \in (-1,1), \\ u(-1) = u(1) = 0. \end{cases} \tag{0.1}$$

u_{xx} will indicate the second-order derivative with respect to x and u_x the first-order derivative. It is easy to see that the exact solution is

$$u(x) = \sin(\pi x).$$

Spectral method:

1) Consider a set $\{\phi_n(x)\}_{n=0}^p$ of basis functions, where

$$\phi_i(x) = P_{n+2}(x) - P_n(x) \tag{0.2}$$

with $P_n(x)$ being Legendre polynomials of order n.

2) Assume that the solution can be approximated as

$$u(x) \approx u_h(x) = \sum_{n=0}^{p} u_n \phi_n(x). \tag{0.3}$$

We substitute $u_h(x)$ to equation (0.1) and require that the residual is orthogonal to the linear space spanned by $\{\phi_n(x)\}_{n=0}^p$, i.e.,

$$\int_{-1}^{1} -u_{h,xx}\phi_m(x)dx = \int_{-1}^{1} \pi^2 \sin(\pi x)\phi_m(x)dx, \quad m = 0, 1, \ldots, p. \tag{0.4}$$

This is also called the Galerkin projection, which results in an equation in terms of the unknown coefficients u_n.

About the Legendre polynomials: The following two formulas will be useful.

1) Three-term recurrence formula:

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \quad P_0(x) = 1, \quad P_1(x) = x. \tag{0.5}$$

2) About the first-order derivative:

$$P_n(x) = \frac{1}{2n+1}P'_{n+1}(x) - \frac{1}{2n+1}P'_{n-1}(x), \quad P_0(x) = P'_1(x). \tag{0.6}$$

Questions:

1) Use the three-term recurrence formula (0.5) to show that any basis function $\phi_n(x)$ satisfies the boundary conditions, i.e.,

$$\phi_n(\pm 1) = 0. \tag{0.7}$$

In other words, the approximation $u_h(x)$ given in equation (0.3) will satisfies the boundary conditions of equation (0.1) automatically. We can then focus on the equation itself.

2) Use integration by part and equation (0.7) to show that equation (0.4) is equivalent to

$$\int_{-1}^{1} u_{h,x}(x)\phi_{m,x}(x)dx = \int_{-1}^{1} \pi^2 \sin(\pi x)\phi_m(x)dx, \quad m = 0, 1, \ldots, p. \tag{0.8}$$

Equation (0.8) is also called the weak form of equation (0.1). Now equation (0.8) can be written in a matrix form:

$$A\mathbf{y} = \mathbf{f}, \tag{0.9}$$

where

$$\begin{aligned} a_{mn} &= \int_{-1}^{1} \phi_{m,x}(x)\phi_{n,x}(x)dx, \\ y_i &= u_i, \\ f_i &= \int_{-1}^{1} \pi^2 \sin(\pi x)\phi_m(x)dx. \end{aligned}$$

3)Construct the linear system (0.9). You can use the following Gauss-Lobatto quadrature:

```
-1.00000000000000e+00   5.26315789473684e-03
-9.80743704893914e-01   3.22371231884888e-02
-9.35934498812666e-01   5.71818021275669e-02
-8.66877978089950e-01   8.06317639961195e-02
-7.75368260952056e-01   1.01991499699451e-01
-6.63776402290311e-01   1.20709227628675e-01
-5.34992864031886e-01   1.36300482358724e-01
-3.92353183713909e-01   1.48361554070917e-01
-2.39551705922986e-01   1.56580102647476e-01
-8.05459372388218e-02   1.60743286387846e-01
 8.05459372388213e-02   1.60743286387846e-01
 2.39551705922986e-01   1.56580102647476e-01
 3.92353183713909e-01   1.48361554070917e-01
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5.34992864031886e-01	1.36300482358724e-01
6.63776402290311e-01	1.20709227628675e-01
7.75368260952056e-01	1.01991499699451e-01
8.66877978089950e-01	8.06317639961197e-02
9.35934498812666e-01	5.71818021275668e-02
9.80743704893914e-01	3.22371231884888e-02
1.00000000000000e+00	5.26315789473684e-03

The first column includes the quadrature points and the second column includes the corresponding integration weights.

4)Use Gauss-Seidel method to solve the linear system (0.9) with a relative tolerance $\varepsilon < 10^{-12}$, where the relative tolerance is defined as

$$\varepsilon = \frac{\|\mathbf{y}^{(k+1)} - \mathbf{y}^{(k)}\|_\infty}{\|\mathbf{y}^{(k+1)}\|_\infty}$$

with k being the iteration step.

Requirements:

1) You should not use any available functions from either Matlab or other programming language to do numerical integration or to solve the linear system. I need your original code.

2) Solve the problem using $p = 3, 4, 5, 6, 7, 8, 9, 10$. Compute the the error

$$\epsilon(p) = \left(\int_{-1}^{1} (\sin(\pi x) - u_h(x))^2 dx \right)^{1/2}$$

and plot the function $\log \epsilon(p)$ in terms of p.