Title: An investigation into the effect of varying type of liquid in a cylindrical shell on its rotational motion along a ramp

Research Question: How does the type of liquid within a hollow cylindrical cylinder shell affect the time it takes to roll down an inclined plane?

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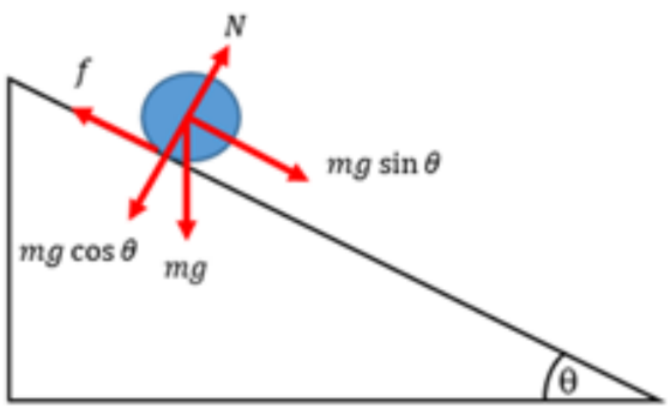
1 Introduction

The purpose of this research report is to answer the following question: *How does the type of liquid within a hollow cylindrical cylinder shell affect the time it takes to roll down an inclined plane?* The objective is to determine a quantitative relationship between the type of viscous liquid in a cylindrical shell and the subsequent time it takes to reach the bottom of the ramp as well as establish a greater conceptual understanding between the two variables of kinematic viscosity, which is a "measure of a fluid's internal resistance to flow under gravitational forces," (Troyer, n.d.) and time in the context of the physical system illustrated in Figure 1.

Figure 1. *Illustration of the forces acting on the center of mass of the liquid-filled hollow cylindrical shell*



Understanding the theoretical relationship between kinematic viscosity and time is done in two steps. First, a numerical relationship between the variables will be established in order to be tested experimentally for validity. Second, as there is no recorded solution to the problem posed in this essay, the experimental results will be compared to the theoretical values in order to check for their reliability and accuracy. Furthermore, to establish the numerical relationship between kinematic viscosity and time and to answer the research question requires delving into so–called ‘convoluted’ integrals and setting dimensionless parameters to determine an analytical solution through establishing viscosity boundary (e.g. liquid exhibiting properties pertaining to being inviscid or having infinite viscosity) conditions in order to model the behavior of liquids that do not fall into either category.

To reduce external factor influence, the research scope has been limited. Particularly, the material of the can as well as the can itself is held constant throughout the experimental process. The reason for this is to omit the potential effects that different materials, specifically with different thermal conductivities, have on a fluid’s flow rate in the cylindrical shell when it is set into motion.

2 Theoretical Background

Viscosity boundary conditions

The first step to deriving an analytical solution to the proposed investigation first requires the assumption that the moment of inertia and mass of the can be omitted as it does not apply to fluids. The reasoning behind this assumption lies in the fact that a fluid is capable of shearing due to the shear stress caused between fluid particles that causes it to move due to fluid viscosity. Therefore, the fluid cannot undergo a fully-rigid rotation along the inclined plane, and thus, the moment of inertia and mass of the can can be discarded in further calculations which would only be relevant in scenarios of fully-rigid body rotation. In the fluid mechanics textbook *Transport Phenomena* authored by Edwin n. Lightfoot, Robert Byron Bird, and Warren E. Stewart released in 1960, they solve the problem of viscous flow near a wall suddenly set in motion, and particularly, of a semi-infinite body of liquid with constant density and viscosity which is bounded by a horizontal surface. Presumably, the fluid and solid (e.g. cylindrical shell) are at rest at $t = 0$, and then are set into motion in the horizontal direction with a velocity V. Their findings show that, at a time t, the shear stress exerted by a fluid can be determined by:

$$\sigma = \mu \frac{V}{\delta(t)}$$

where μ is the fluid viscosity and δ is the boundary layer thickness in effect, which is the distance from a solid's surface to the point where the fluid velocity reaches 99% of the free stream velocity according to Munson, Okiishi, Huebsch, and Rothmayer, and can be quantized through:

$$\delta = \sqrt{\pi v t}$$

where v is the kinematic viscosity which is further given by:

$$v = \frac{\mu}{\rho}$$

where ρ represents the density of the fluid. Therefore, as Lightfoot, Bird, and Stewart concluded, the torque, τ, imposed on fluid on the inside surface of the can can be determined through the following expression:

$$\tau = 2\pi R^2 L \sigma = 2\pi R^2 L \frac{\mu V}{\sqrt{\pi v t}} = 2\rho R^2 L \sqrt{\pi v} \frac{V}{\sqrt{t}} \qquad (1)$$

where R is the inner radius of the can. However, the solution the physical system posed requires this formula to be adjusted to account for the fact that the torque imposed by the shearing fluid on the inside surface of the can is a linear function of the tangential inside surface velocity can. Equation 1 does not take into account the aspect of time and thus it can be updated to acknowledge that the rotation of the can is variable in time through linear superposition that is achieved through a convoluted integration method. This method was specifically chosen as this type of integration utilizes a "dummy" variable (e.g. ξ) in order to solve a differential equation.

$$\tau(t) = 2\rho R^2 L\sqrt{\pi v} \cdot \int_0^t \frac{V'(\xi)}{\sqrt{t-\xi}} d\xi \qquad (2)$$

where $V' = \frac{dV}{d\xi}$ and ξ is a time variable of integration. Furthermore, it should be noted that:

$$V'(xi) = R\alpha(\xi) \qquad (3)$$

where α is the angular acceleration of the can. Therefore, through the combination of these equations, it is revealed that:

$$\tau(t) = 2\rho R^3 L\sqrt{\pi v} \cdot \int_0^t \frac{\alpha(\xi)}{\sqrt{t-\xi}} d\xi \qquad (4)$$

Moving on, it is also necessary to take into account the moment balance of the can in order to derive the formula for frictional force through balancing the torque in the physical system. According to *Engineering Mechanics: Statics and Dynamics* written by James L. Meriam and L. G. Kraige and released in 2013, by considering the balance of forces and torques on a can rolling down an inclined plane, they determined that:

$$FR_0 - \tau = I_c\alpha \qquad (5)$$

where R_0 is the outside radius of the can. Further substitution as well as solving for the frictional force reveals that:

$$I_c = M_c \frac{R_0^2 + R^2}{2} \qquad (6)$$

where M_c represents the mass of the can. Furthermore, if we substitute prior equations we find that:

$$F = 2\rho\frac{R^3}{R_0}L\sqrt{\pi v} \cdot \int_0^t \frac{\alpha(\xi)}{\sqrt{t-\xi}}d\xi + M_c\frac{R_0^{\ 2}+R^2}{2}\alpha \qquad (7)$$

Moreover, this formula can be further simplified when understanding that the angular acceleration α is kinematically related to the acceleration of the center of mass of the can "a" through:

$$\alpha = \frac{a}{R_0} \qquad (8)$$

Therefore, substituting Eq. 8 into Eq. 7 and simplifying further yields:

$$2\rho\frac{R^3}{R_0^{\ 2}}L\sqrt{\pi v} \cdot \int_0^t \frac{\alpha(\xi)}{\sqrt{t-\xi}}d\xi + M_c\frac{1+(R/R_0)}{2}a(t)$$

$$= M_L\kappa^2\left(\frac{2\sqrt{\pi v}}{\pi R}\right)\int_0^t \frac{\alpha(\xi)}{\sqrt{t-\xi}}d\xi + M_c\frac{(1+\kappa^2)}{2}a(t) \qquad (9)$$

where M_L is the mass of the liquid in the can and $\kappa = \left(\frac{R}{R_0}\right)$. However, it has been stated and explained that the variance in mass can be neglected. Similarly, as the inner and outer radii of the can used in the experimental stage of this investigation are approximately equal, Eq. 9 can be simplified as:

$$M_L\left(\frac{2\sqrt{\pi v}}{\pi R}\right)\int_0^t \frac{\alpha(\xi)}{\sqrt{t-\xi}}d\xi \qquad (10)$$

Additionally, for the case where the outer boundary layer is thin and otherwise negligible in the rotating fluid, Eq. 10 can be reduced further to yield:

$$F = \frac{4}{\sqrt{\pi}}M_L\frac{\sqrt{vt}}{R}a \qquad (11)$$

Therefore, it can be deduced that the force balance equation can be written as:

$$M_L gsin\alpha - F = M_L a$$

or

$$a(t) = \frac{dv}{dt} = \frac{gsin\alpha}{1+\frac{4}{\sqrt{\pi}}\frac{\sqrt{vt}}{R}} \approx gsin\alpha\left(1 - \frac{4}{\sqrt{\pi}}\frac{\sqrt{vt}}{R}\right) \qquad (12)$$

Then, it is possible to take the integral of this function in order to get a velocity function:

$$v = gsin\alpha(1 - \frac{8}{3\sqrt{\pi}}\frac{\sqrt{vt}}{R})t \quad (13)$$

Furthermore, integrating it once again will provide a function for the distance:

$$L \approx gsin\alpha(1 - \frac{32}{15\sqrt{\pi}}\frac{\sqrt{vt}}{R})\frac{t^2}{2} \qquad (14)$$

which to the same level of approximation shows that:

$$L(1 + \frac{32}{15\sqrt{\pi}}\frac{\sqrt{vt}}{R}) \approx (gsin\alpha)\frac{t^2}{2} \qquad (15)$$

The first approximation to the solution to the physical system in terms of the time taken to reach the bottom of the ramp for a completely inviscid solution is given by:

$$t \approx t_0 = \sqrt{\frac{2L}{gsin\alpha}} \qquad (16)$$

However, an arguably more accurate solution can be found through substituting this expression for t_0 into the term in parenthesis and subsequently solving for t:

$$t \approx t_0\sqrt{1 + \frac{16}{15}\sqrt{\frac{vt_0}{\pi R^2}}} \qquad (17)$$

This equation thereby represents the behavior in the physical system described if and only if the mass and the moment of inertia are treated as negligible as done in the investigation, and only in the limit of $\frac{16}{15}\sqrt{\frac{vt_0}{\pi R^2}} << 1$. This function will act as the range of times with the viscosity boundary conditions of a fluid that is completely inviscid and one that has properties of infinite viscosity and will be cross–checked with the actual experimental results to ascertain its validity and accuracy.

Dimensionless parameters

Asymptotic solution at long times