Title: An investigation into the effect of varying type of liquid in a cylindrical shell on its rotational motion along a ramp

Research Question: How does the type of liquid within a hollow cylindrical cylinder shell affect the time it takes to roll down an inclined plane?

IB Physics Extended Essay

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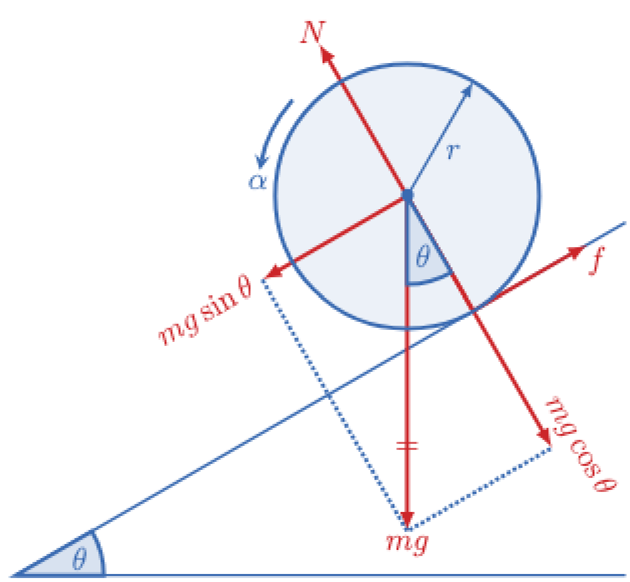
Table of Contents

1 Introduction

The purpose of this research report is to answer the following question: *How does the type of liquid within a hollow cylindrical shell affect the time it takes to roll down an inclined plane?* The objective is to determine a quantitative relationship between the type of viscous liquid in a cylindrical shell and the subsequent time it takes to reach the bottom of the ramp as well as establish a greater conceptual understanding between the variables of kinematic viscosity, which is a "measure of a fluid's internal resistance to flow under gravitational forces," (Troyer, n.d.) and time in the context of the physical system illustrated in **Figure 1**.

Figure 1. *Illustration of the forces acting on the center of mass of the liquid-filled hollow cylindrical shell.Note. Singh, J. (2020a, March 2). NAEST 2015 screening test solution: Cylinder rolling down an inclined plane. NAEST 2015 Screening Test Solution | Cylinder Rolling Down an Inclined Plane.*



Understanding the theoretical relationship between kinematic viscosity and time is done in two steps. First, a mathematical relationship between the variables will be established in order to be tested experimentally for validity. Second, as there is no recorded solution to the problem posed in this essay, the experimental results will be compared to the theoretical values in order to check for their reliability and accuracy. Furthermore, to establish the mathematical relationship between kinematic viscosity and time and to answer the research question requires delving into so–called 'convoluted' integrals and setting dimensionless parameters to determine an analytical solution through establishing viscosity boundary (e.g. liquid exhibiting properties pertaining to being inviscid or having infinite viscosity) conditions in order to model the behavior of liquids that do not fall into either category.

To reduce external factor influence, the research scope has been limited. Particularly, the material of the can as well as the can itself is held constant throughout the experimental process. The reason for this is to omit the potential effects that different materials, specifically with different thermal conductivities, have on a fluid's flow rate in the cylindrical shell when it is set into motion.

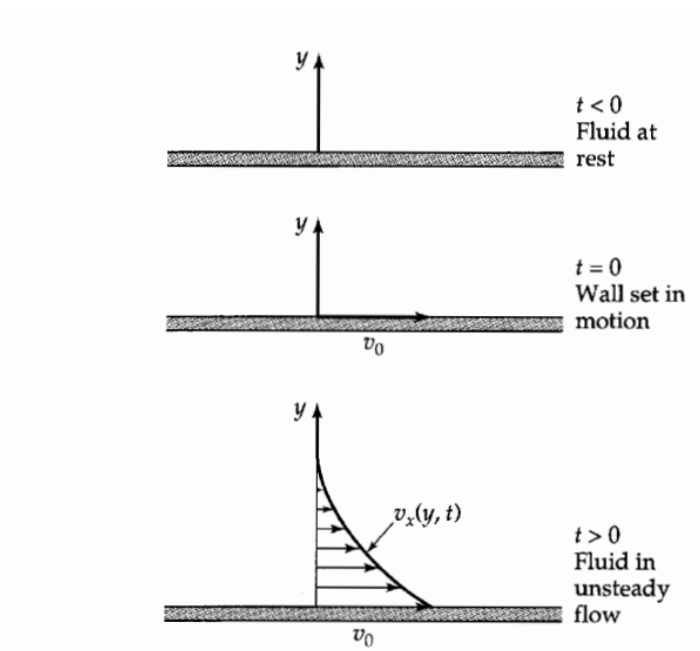
2 Theoretical Background

Viscosity boundary conditions

The first step to deriving an analytical solution to the proposed investigation first requires the assumption that the moment of inertia and mass of the cylindrical can be omitted as it does not apply to fluids. The reasoning behind this assumption lies in the fact that a fluid is capable of shearing, which means that it is capable of flowing as well as changing shape (Princeton) due to the shear stress caused between fluid particles that results it to move due to fluid viscosity.

Therefore, the fluid cannot undergo a fully-rigid rotation along the inclined plane, and thus, the moment of inertia and mass of the can can be discarded in further calculations which would only be relevant in scenarios of fully-rigid body rotation. In the fluid mechanics textbook *Transport Phenomena* authored by Edwin n. Lightfoot, Robert Byron Bird, and Warren E. Stewart released in 1960, they solve the problem of viscous flow near a wall suddenly set in motion, and particularly, of a semi-infinite body of liquid with constant density and viscosity which is bounded by a horizontal surface of which a diagram is shown in **Figure 2**.

Figure 2. *Visual representation of the problem regarding viscous flow of a fluid near a wall suddenly set in motion. Note.* Bird, R. B., Stewart, W. coautor, & Lightfoot, E. n. (1960). Transport phenomena. J. Wiley.



Presumably, the fluid and solid (e.g. cylindrical shell) are at rest at $t < 0$, and then are set into motion at $t = 0$ in the horizontal direction with a velocity V. Their findings show that, at a time t, the shear stress exerted by a fluid can be determined by:

$$\sigma = \mu \frac{V}{\delta(t)}$$

where μ is the fluid viscosity and δ is the boundary layer thickness in effect, which is the distance from a solid's surface to the point where the fluid velocity reaches 99% of the free stream velocity according to Munson, Okiishi, Huebsch, and Rothmayer, and can be quantified through:

$$\delta = \sqrt{\pi v t}$$

where v is the kinematic viscosity which is further given by:

$$v = \frac{\mu}{\rho}$$

where ρ represents the density of the fluid. Therefore, the torque, τ, imposed on fluid on the inside surface of the can can be determined through the following expression:

$$\tau = 2\pi R^2 L\sigma = 2\pi R^2 L \frac{\mu V}{\sqrt{\pi v t}} = 2\rho R^2 L \sqrt{\pi v} \frac{V}{\sqrt{t}} \qquad (1)$$

where R is the inner radius of the can. However, the solution to the primary research question requires this formula to be adjusted to account for the fact that the torque imposed by the shearing fluid on the inside surface of the can is a linear function of the tangential inside surface velocity can. Equation 1 does not take into account the aspect of time and thus it can be updated to acknowledge that the rotation of the can is variable in time through linear superposition that is achieved through a convolution integration method. This method was specifically chosen as this type of integration utilizes a "dummy" variable (e.g. ξ) in order to solve a differential equation (Wang).

$$\tau(t) = 2\rho R^2 L\sqrt{\pi v} \cdot \int_0^t \frac{V'(\xi)}{\sqrt{t-\xi}} d\xi \qquad (2)$$

where $V' = \frac{dV}{d\xi}$ and ξ is a time variable of integration. Furthermore, it should be noted based on the fact that there exists a relationship between the tangential inside surface velocity and the angular acceleration of the can such that:

$$V'(xi) = R\alpha(\xi) \qquad (3)$$

where α is the angular acceleration of the can. Therefore, through the combination of these equations, it is revealed that:

$$\tau(t) = 2\rho R^3 L\sqrt{\pi v} \cdot \int_0^t \frac{\alpha(\xi)}{\sqrt{t-\xi}} d\xi \qquad (4)$$

Moving on, it is also necessary to take into account the moment balance of the can in order to derive the formula for frictional force through balancing the torque in the physical system. According to *Engineering Mechanics: Statics and Dynamics* written by James L. Meriam and L. G. Kraige and released in 2013, by considering the balance of forces and torques on a can rolling down an inclined plane, they determined that:

$$FR_0 - \tau = I_c\alpha \qquad (5)$$

where R_0 is the outside radius of the can and I_c is the moment of inertia of the object. Further substitution as well as solving for the frictional force reveals that:

$$I_c = M_c\frac{R_0^2 + R^2}{2} \qquad (6)$$

where M_c represents the mass of the can. Furthermore, if we substitute prior equations we find that:

$$F = 2\rho\frac{R^3}{R_0}L\sqrt{\pi v}\cdot\int_0^t\frac{\alpha(\xi)}{\sqrt{t-\xi}}d\xi + M_c\frac{R_0^{\ 2}+R^2}{2}\alpha \qquad (7)$$

Moreover, this formula can be further simplified when understanding that the angular acceleration α is kinematically related to the acceleration of the center of mass of the can "a" through:

$$\alpha = \frac{a}{R_0} \qquad (8)$$

Therefore, substituting Eq. 8 into Eq. 7 and simplifying further yields:

$$2\rho\frac{R^3}{R_0^{\ 2}}L\sqrt{\pi v}\cdot\int_0^t\frac{\alpha(\xi)}{\sqrt{t-\xi}}d\xi + M_c\frac{1+(R/R_0)}{2}a(t)$$

$$= M_L\kappa^2\left(\frac{2\sqrt{\pi v}}{\pi R}\right)\int_0^t\frac{\alpha(\xi)}{\sqrt{t-\xi}}d\xi + M_c\frac{(1+\kappa^2)}{2}a(t) \qquad (9)$$

where M_L is the mass of the liquid in the can and $\kappa = \left(\frac{R}{R_0}\right)$. However, it has been stated and explained that the variance in mass can be neglected. Similarly, as the inner and outer radii of the can used in the experimental stage of this investigation are approximately equal, Eq. 9 can be simplified as:

$$F = M_L\left(\frac{2\sqrt{\pi v}}{\pi R}\right)\int_0^t\frac{\alpha(\xi)}{\sqrt{t-\xi}}d\xi \qquad (10)$$

Additionally, for the case where the outer boundary layer is thin and otherwise negligible in the rotating fluid, Eq. 10 can be reduced further to yield:

$$F = \frac{4}{\sqrt{\pi}}M_L\frac{\sqrt{vt}}{R}a \qquad (11)$$

Therefore, it can be deduced that the force balance equation can be written as:

$$M_L g sin\alpha - F = M_L a$$

or

$$a(t) = \frac{dv}{dt} = \frac{gsin\alpha}{1+\frac{4}{\sqrt{\pi}}\frac{\sqrt{\nu t}}{R}} \approx gsin\alpha\left(1 - \frac{4}{\sqrt{\pi}}\frac{\sqrt{\nu t}}{R}\right) \qquad (12)$$

Then, it is possible to take the integral of this function in order to get a velocity function:

$$v = gsin\alpha(1 - \frac{8}{3\sqrt{\pi}}\frac{\sqrt{\nu t}}{R})t \quad (13)$$

Furthermore, integrating it once again will provide a function for the distance:

$$L \approx gsin\alpha(1 - \frac{32}{15\sqrt{\pi}}\frac{\sqrt{\nu t}}{R})\frac{t^2}{2} \qquad (14)$$

which to the same level of approximation shows that:

$$L(1 + \frac{32}{15\sqrt{\pi}}\frac{\sqrt{\nu t}}{R}) \approx (gsin\alpha)\frac{t^2}{2} \qquad (15)$$

The first approximation to the solution to the physical system in terms of the time taken to reach the bottom of the ramp for a completely inviscid solution is given by:

$$t \approx t_0 = \sqrt{\frac{2L}{gsin\alpha}} \qquad (16)$$

However, an arguably more accurate solution can be found through substituting this expression for t_0 into the term in parenthesis and subsequently solving for t:

$$t \approx t_0(1 + \frac{16}{15}\sqrt{\frac{\nu t_0}{\pi R^2}}) \qquad (17)$$

This equation thereby represents the behavior in the physical system described if and only if the mass and the moment of inertia are treated as negligible as done in the investigation, and only in the limit of $\frac{16}{15}\sqrt{\frac{\nu t_0}{\pi R^2}} << 1$. This function will act as the range of times with the viscosity boundary conditions of a fluid that is completely inviscid and one that has properties of infinite viscosity and will be cross–checked with the actual experimental results to ascertain its validity and accuracy.

Asymptotic solution at long times

As the can accelerates, the fluid inside the can exhibits a non-uniform angular velocity distribution, with the angular velocity at the can surface exceeding that of the fluid in the interior. This discrepancy in angular velocity generates a shear stress profile within the fluid, which in turn influences the can's rotational motion. This situation bears a striking resemblance to the transient heat conduction problem in a solid cylinder subjected to a constant heat flux at its surface. In the heat transfer scenario, the temperature inside the cylinder initially lags behind the surface temperature, leading to a radial temperature gradient and a corresponding radial heat flux profile. Over time, the temperature at each radial location within the cylinder increases linearly with time. However, this linear increase is accompanied by a time-independent radial temperature profile that ensures a uniform rate of temperature rise at all radial locations. This analogy between the fluid-filled can and the solid cylinder under transient heat conduction highlights the role of transport phenomena in influencing the dynamics of rotating objects in viscous fluids. The shear stress profile in the fluid, analogous to the radial temperature gradient in the solid cylinder, plays a crucial role in dissipating energy and affecting the rotational motion of the can.

Therefore, if we utilize this understanding, it is possible to determine the asymptotic solution at long time intervals to the two principle dimensionless equations 18 and 19 which are derived in **Appendix A**. From there, we are able to deduce that, at long times, the angular velocity profile in the fluid approaches:

$$\varpi = \frac{2}{3}\bar{t} + \frac{1}{12}\bar{r}^{2} \qquad (20)$$

where ϖ, $\bar{t}$, and $\bar{r}$ are dimensionless parameters, which allow us to be able to compare the relative physical effects of each term in an expression. The aforementioned dimensionless parameters may be expressed as:

$$\varpi = \frac{\omega\nu}{Rg\sin\alpha} \qquad (21)$$

$$\bar{t} = \frac{\nu t}{R^2} \qquad (22)$$

$$\bar{r} = \frac{r}{R} \qquad (23)$$

Therefore, Eq. 20 and the Equations introduced from 21-23 can be combined in order to form an expression that will thereby function as the asymptotic solution at long times to the rolling cylinder problem posed in the report:

$$\varpi = \frac{2}{3}\frac{g\sin\alpha}{R}t + \frac{g\sin\alpha}{R}\frac{r^2}{12\nu} \qquad (24)$$

$$\frac{d\omega}{dt} = \frac{2}{3}\frac{g\sin\alpha}{R}$$

$$a = \frac{2}{3}g\sin\alpha \qquad (25)$$

Eq. 25 demonstrates that a cylinder that is filled entirely with a highly viscous fluid nearing infinity and is subject to motion along an inclined plane will have the same linear acceleration as a cylinder that was entirely solid. In that same regard, the inviscid case of acceleration can be analytically determined to be:

$$a = g\sin\alpha \qquad (26)$$

Furthermore, it can be realized that the solution for the asymptotic long time viscous acceleration is $\frac{3}{2}$ times the inviscid case of acceleration. Therefore, at long times, the time to roll down the ramp increases by a factor of $\sqrt{\frac{3}{2}}$. As a result, since the time for inviscid times is 1.29 seconds as determined through $t = \sqrt{\frac{2L}{g sin\alpha}}$, where L is the length of the ramp and the angle α is 10.03 ± 2° as experimentally determined within the report through the use of the physical dimensions of the ramp, then at long times it is $\sqrt{\frac{3}{2}} * 1.29 = 1.57$ seconds.

3 Variables

Independent variables

The independent variable is simply the liquid within the can.

Dependent variables

In this experiment, the dependent variables explored are the linear acceleration of the can and the time it takes for the can to reach the bottom of the ramp.

Controlled variables

To ensure that the motion of the can is solely affected by the type of fluid inside the can, several variables were kept constant throughout the experiment. The angle of the ramp (α) was maintained at a fixed value to eliminate any influence from variations in the ramp's incline. Similarly, the mass of the can (m) was kept constant to prevent changes in mass from affecting the can's acceleration. Additionally, the dimensions of the can, including its radius (R) and length (L), were held constant to maintain consistent frictional interactions between the can and the ramp. Furthermore, the temperature of the fluid was kept constant to ensure that its viscosity remained unchanged, preventing temperature fluctuations from influencing the fluid's behavior.

4 Methodology

Apparatus

a. Wooden ramp (1.454 ± 0.001 m)

c. 30cm Ruler (for precise measurements)

d. Stopwatch (for timing the can's descent)

e. Various liquids: sunflower cooking oil, automated transmission fluid, water, and molasses honey (for experimental trials with different liquids)

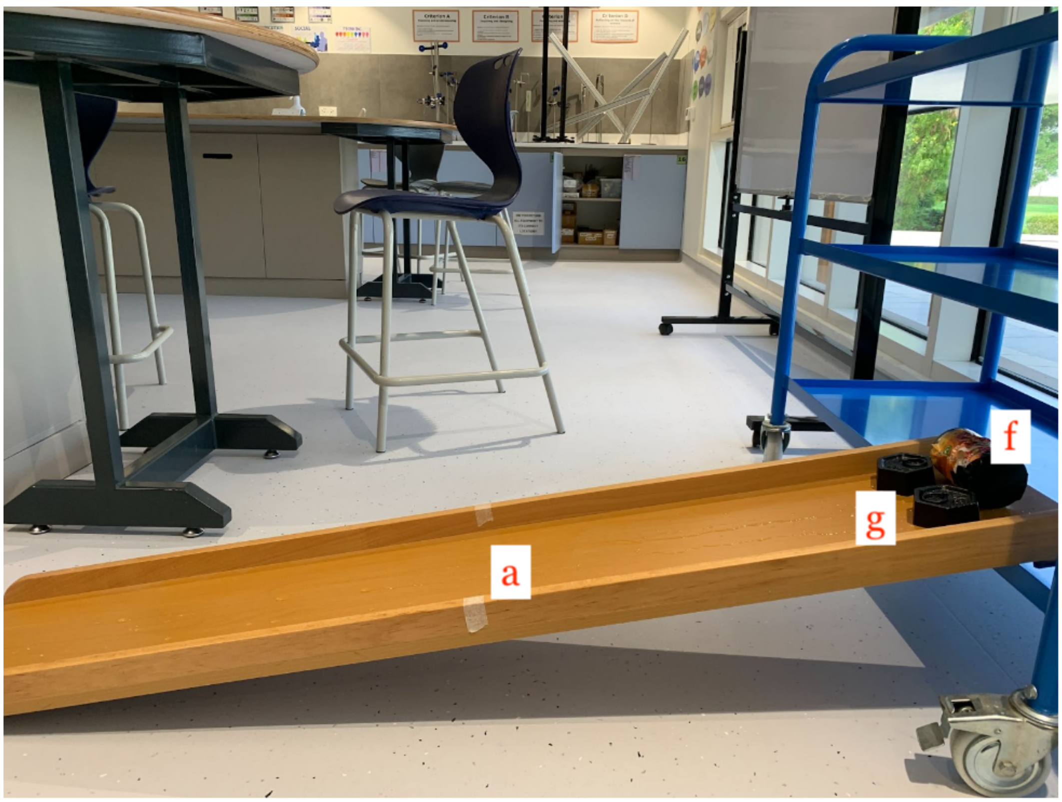
f. Cylindrical hollow can (for the experiment's main object of study)

g. 1kg hexagonal weight (to maintain cylinder's initial point of rest) (2)

h. Duct tape (1)

i. Digital mass scale (1)

Photograph taken by candidate.



Procedure

- Set up the wooden ramp on a stable surface, ensuring it is secure and stationary.
- Use a leveling tool to ensure the ramp is properly aligned and has a consistent angle of inclination (e.g., 10.03 ± 2°). Adjust if needed.

- Place the cylindrical hollow can at the top of the ramp, aligning it with the center of the ramp and perpendicular to the surface.
- Begin with the can filled with one type of liquid (e.g., sunflower cooking oil).
- Hold the can at the top of the ramp, ensuring it is still and not rotating.
- Release the can gently from rest, allowing it to roll down the ramp without any additional force.
- Start the stopwatch as soon as the can is released and stop it when the can reaches a designated point on the ramp. Use a visual marker or a distinct feature on the ramp to mark the point.
- Repeat this process six times for each liquid type.

5 Experimental Findings

Raw data tables

Table 1: Dimensions of Hollow Cylindrical Can

Dimensions of can	**Trial 1**	**Trial 2**	**Trial 3**
Radius ± 0.0005 (m)	3.50	3.45	3.48
Height ± 0.0005 (m)	11.30	11.28	11.35

Table 2: Mass of Each Liquid With and Without the Duct Tape Covering (where mass of can is negligible)

liquid Type	**Trials**	**Mass of liquid + tape (g)**	**Mass of liquid (g)**
Automated transmission liquid dextrin			

	Trial 1	334	333
	Trial 2	334	333
	Trial 3	334	333
Sunflower oil			
	Trial 1	414	413
	Trial 2	414	413
	Trial 3	414	413
Water			
	Trial 1	445	444
	Trial 2	445	444
	Trial 3	445	444
molasses honey			
	Trial 1	550	549
	Trial 2	550	549
	Trial 3	550	549

Table 3: Time taken to reach bottom of ramp in seconds for each liquid fully-filled into a hollow cylindrical shell.

Liquid Type	**Trials**	**Time taken to reach bottom of ramp** ± 0.01 **(s)**
Automated transmission liquid dextrin		
	Trial 1	1.40
	Trial 2	1.41
	Trial 3	1.38
Sunflower oil		

	Trial 1	1.45
	Trial 2	1.42
	Trial 3	1.43
Water		
	Trial 1	1.50
	Trial 2	1.47
	Trial 3	1.50
Molasses honey		
	Trial 1	1.30
	Trial 2	1.32
	Trial 3	1.34

Table 4: Measurements of the diagonal length (hypotenuse) and height of the wooden ramp in meters

Trials	Hypotenuse (diagonal length of ramp) ± 0.05 (cm)	Height ± 0.05 (cm)
Trial 1	145.4	25.4
Trial 2	144.8	25.8
Trial 3	144.5	25.0

Processed data tables

Table 5:

Average height ± 0.0015 (m)	Average radius ± 0.0015 (m)	Average volume of the can ± 0.0060 (m^3)
11.31	3.4712	427.83

Table 6: Processed values for density, subsequent viscosity, μ, in poise, and kinematic viscosity, ν, in $\frac{cm^2}{s}$.

	Density (ρ) ± 0.6 $\frac{g}{cm^3}$	Approximated viscosity (μ) in poise	Kinematic viscosity (ν)
Water	1.0	0.01	0.01
Sunflower oil	0.92	0.49	0.53
Molasses honey	1.45	120.00	82.75
Transmission fluid	0.87	1.20	1.37

Table 7: Processed values for average time taken to reach bottom of ramp for different liquids within the can.

	Average time taken to reach bottom of ramp (s) ± 0.03 seconds
Water	1.49
Sunflower oil	1.43
Molasses honey	1.32
Transmission fluid	1.39

Graphical representation of the data

The primary graphical representation of data in this report will be in the context of the formula derived in the theoretical background. Particularly, Equation 17, which presents an expression for short-time behavior:

$$t = t_0 \left(1 + \frac{16}{15}\sqrt{\frac{\nu t_0}{\pi R^2}}\right)$$

Therefore, it is sufficient to plot $\frac{t}{t_0}$ against $\sqrt{\frac{\nu t_0}{\pi R^2}}$ in order to describe the behavior of the various liquids filled to fit the volume of the cylindrical can.

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