Correct TE wave solution in a rectangular waveguide for a GGT coordinate system

First some quick definitions and explanation of the setup:

Axis / origin:

Call the waveguide axis the z-axis, the x-axis is perpendicular to two walls, and the y-axis perpendicular to the remaining walls. Define the origin to be on a corner, the width in the x direction to be a, and the width in the y direction to be b.

I will refer to the components of the fields using subscripts (ie $\vec{E} = \hat{x}E_x + \hat{y}E_y + \hat{z}E_z$). Note that because we are interested in the TE mode, $E_z = 0$.

In SR (or the 'special frame' of GGT), the cutoff frequency for the mn mode is

$$\omega_{mn}^2 = c^2(\frac{m\pi}{a})^2 + c^2(\frac{n\pi}{b})^2. \tag{1}$$

I will use this as the definition of ω_{mn}, and use ω_c to refer to the cutoff frequency in a GGT frame (as per Gagnon's notation).

As has been proved multiple times in a very straight forward manner, ω_{mn} will also be the cutoff frequency for GGT. Since some individuals refuse to see the forest unless each tree is numbered, this is the "long method" of proving that the frequency does indeed stay the same. That is, I will solve Maxwell's equations in the moving GGT frame.

So, here we go...

We are interested in the travelling waves, so we can assume a solution of the form:

$$\vec{E} = \vec{E}(x,y)\exp(ikz - i\omega t) \tag{2}$$
$$\vec{B} = \vec{B}(x,y)\exp(ikz - i\omega t) \tag{3}$$

Our boundary conditions are

$$E_{\parallel} = 0, \quad B_{\perp} = 0 \tag{4}$$

Where parallel refers to components parallel at a waveguide wall, and perpendicular refers to the the component perpendicular at a waveguide wall.

Along with

$$E_z = 0 \tag{5}$$

and Maxwell's equations for a GGT frame

$$\vec{\nabla} \cdot [\epsilon_0 \vec{E} - \epsilon_0 \vec{v} \times \vec{B} + \epsilon_0 c^{-2} \vec{v} \times (\vec{v} \times \vec{E})] = \rho \tag{6}$$
$$\vec{\nabla} \times [\mu_0^{-1}\vec{B} - \epsilon_0 \vec{v} \times \vec{E}] = j + \frac{\partial}{\partial t}[\epsilon_0 \vec{E} - \epsilon_0 \vec{v} \times \vec{B} + \epsilon_0 c^{-2} \vec{v} \times (\vec{v} \times \vec{E})] \tag{7}$$
$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} \tag{8}$$
$$\vec{\nabla} \cdot \vec{B} = 0 \tag{9}$$

this specifies the entire problem. So let's plug and chug through it.

Let's look at how $\vec{E}$ and $\vec{B}$ are related using $-\frac{\partial}{\partial t}\vec{B} = \vec{\nabla} \times \vec{E}$

$$-\frac{\partial B_x}{\partial t} = (\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}) = -ikE_y \tag{10}$$

$$-\frac{\partial B_y}{\partial t} = (\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}) = ikE_x \tag{11}$$

$$-\frac{\partial B_z}{\partial t} = (\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}) \tag{12}$$

Since we already know the time dependance of B, this yields

$$B_x = \frac{-k}{\omega} E_y \tag{13}$$

$$B_y = \frac{k}{\omega} E_x \tag{14}$$

$$B_z = -\frac{i}{\omega}\frac{\partial E_y}{\partial x} + \frac{i}{\omega}\frac{\partial E_x}{\partial y} \tag{15}$$

Applying the boundary condition $B_\perp = 0$ we find:

$$B_x(y=0) = B_x(y=b) = 0 \quad \Rightarrow \quad E_y(y=0) = E_y(y=b) = 0 \tag{16}$$

$$B_y(x=0) = B_y(x=a) = 0 \quad \Rightarrow \quad E_x(x=0) = E_x(x=a) = 0 \tag{17}$$

Note that $\vec{B}$ is now completely specified in terms of $\vec{E}$ and all boundary conditions on $\vec{B}$ have been applied. So we are done with the magnetic field.

Apply the boundary condition $E_\parallel = 0$

$$E_x(y=0) = E_x(y=b) = 0 \tag{18}$$

$$E_y(x=0) = E_y(x=a) = 0 \tag{19}$$

Now we are done with the boundary conditions. If we evalute $\vec{\nabla} \times (\vec{\nabla} \times \vec{E})$, we find:

$$[\nabla^2 + 2(\frac{\vec{v}}{c} \cdot \vec{\nabla})\frac{1}{c}\frac{\partial}{\partial t} - (1 - \frac{v^2}{c^2})\frac{1}{c^2}\frac{\partial^2}{\partial t^2}]\vec{E} = 0 \tag{20}$$

Looking at just the E_x component, and assuming a seperable solution

$$E_x = X(x)Y(y)\exp(ikz - i\omega t) \tag{21}$$

Evaluating the wave equation we find:

$$-k^2XY + \frac{\partial^2 X}{\partial x^2}Y + X\frac{\partial^2 Y}{\partial y^2} + 2(\frac{v_x}{c}\frac{\partial X}{\partial x}Y + \frac{v_y}{c}X\frac{\partial Y}{\partial y} + \frac{v_z}{c}ikXY)\frac{-i\omega}{c} - (1 - \frac{v^2}{c^2})\frac{-\omega^2}{c^2} = 0 \tag{22}$$

Grouping terms and dividing by XY to make the situation more obvious results in

$$[-k^2 + 2\frac{v_z}{c}\frac{\omega}{c}k + (1 - \frac{v^2}{c^2})\frac{\omega^2}{c^2}] + \frac{1}{X}[\frac{\partial^2}{\partial x^2} - i2\frac{v_x}{c}\frac{\omega}{c}\frac{\partial}{\partial x}]X + \frac{1}{Y}[\frac{\partial^2}{\partial y^2} - i2\frac{v_y}{c}\frac{\omega}{c}\frac{\partial}{\partial y}]Y = 0 \tag{23}$$

Because this must be true for all x, y in the waveguide, $\frac{1}{X}[\frac{\partial^2}{\partial x^2} - i2\frac{v_x}{c}\frac{\omega}{c}\frac{\partial}{\partial x}]X$ can't depend on x and $\frac{1}{Y}[\frac{\partial^2}{\partial y^2} - i2\frac{v_y}{c}\frac{\omega}{c}\frac{\partial}{\partial y}]Y$ can't depend on y. Therefore:

$$\frac{1}{X}[\frac{\partial^2}{\partial x^2} - i2\frac{v_x}{c}\frac{\omega}{c}\frac{\partial}{\partial x}]X = -A_x^2 \tag{24}$$

$$\frac{1}{Y}[\frac{\partial^2}{\partial y^2} - i2\frac{v_y}{c}\frac{\omega}{c}\frac{\partial}{\partial y}]Y = -A_y^2 \tag{25}$$

where A_x and A_y are just some constants.

We know that each equation has two linearly independent solutions of the form $X(x) = \exp(irx)$. Plugging this into the differential equation yields

$$-r^2 + 2\frac{v_x}{c}\frac{\omega}{c}r = -A_x^2 \tag{26}$$

$$r_\pm = \frac{v_x}{c}\frac{\omega}{c} \pm \sqrt{(\frac{v_x}{c}\frac{\omega}{c})^2 + A_x^2} \tag{27}$$

So in the general case,

$$X(x) = C_1 \exp(r_+ x) + C_2 \exp(r_- x) \tag{28}$$

Applying the boundary conditions on E_x we see $X(0) = 0$ and thus $C_2 = -C_1$. Thus

$$X(x) = C_1 \exp(ix\frac{v_x}{c}\frac{\omega}{c})[\exp(ix\sqrt{(\frac{v_x}{c}\frac{\omega}{c})^2 + A_x^2}) - \exp(-ix\sqrt{(\frac{v_x}{c}\frac{\omega}{c})^2 + A_x^2})] \tag{29}$$

Now applying the boundary condition $X(a) = 0$,

$$\exp(ia\sqrt{(\frac{v_x}{c}\frac{\omega}{c})^2 + A_x^2}) = \exp(-ia\sqrt{(\frac{v_x}{c}\frac{\omega}{c})^2 + A_x^2}) \tag{30}$$

$$ia\sqrt{(\frac{v_x}{c}\frac{\omega}{c})^2 + A_x^2} = -ia\sqrt{(\frac{v_x}{c}\frac{\omega}{c})^2 + A_x^2} + i2\pi m \tag{31}$$

where m is an integer. Thus:

$$-A_x^2 = (\frac{v_x}{c}\frac{\omega}{c})^2 - (\frac{m\pi}{a})^2. \tag{32}$$

And similarly

$$-A_y^2 = (\frac{v_y}{c}\frac{\omega}{c})^2 - (\frac{n\pi}{b})^2. \tag{33}$$

Returning to eq (23) and plugging this in, we find

$$[-k^2 + 2\frac{v_z}{c}\frac{\omega}{c}k + (1 - \frac{v^2}{c^2})\frac{\omega^2}{c^2}] + (\frac{v_x}{c}\frac{\omega}{c})^2 - (\frac{m\pi}{a})^2 + (\frac{v_y}{c}\frac{\omega}{c})^2 - (\frac{n\pi}{b})^2 = 0 \tag{34}$$

$$-k^2 + 2\frac{v_z}{c}\frac{\omega}{c}k + (1 - \frac{v_z^2}{c^2})\frac{\omega^2}{c^2} - \frac{\omega_{mn}^2}{c^2} = 0 \tag{35}$$

Before continuing any further, note that v_x and v_y are treated equally here, unlike in Gagnon's solution. My roommate and I proposed an explanation for how Gagnon made this mistake (and my roommate found out how to combine some errors to entirely reproduce Gagnon's incorrect result), but you are free to speculate where their error actually occurs. The point is, it is apparrent Gagnon made the largest of their errors before this point.

Solving for k,

$$k = \frac{v_z}{c}\frac{\omega}{c} \pm \sqrt{(\frac{v_z}{c}\frac{\omega}{c})^2 + (1 - \frac{v_z^2}{c^2})\frac{\omega^2}{c^2} - \frac{\omega_{mn}^2}{c^2}}. \tag{36}$$

To make it even more obvious, simplifying

$$k = \frac{v_z}{c}\frac{\omega}{c} \pm \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}. \tag{37}$$

It is clear that the minimum ω (the cutoff frequency) is indeed still $\omega_c = \omega_{mn}$ in GGT. Also notice something else. The minimum ω does NOT correspond to $k = 0$ (this is another of Gagnon's mistakes). This makes sense because in the SR frame, at the cutoff frequency we have $k = 0$ and thus every point agrees on the phase (simultaneity). This should obviously not be true in the GGT frame (which has a different simultaneity convention).

In general, the minimum ω is the smallest frequency before k becomes imaginary (a decaying wave). In SR, this is indeed the same as setting $k = 0$, but not in GGT.

And now we come to a close.