Global Internal Symmetries, Local Gauge Invariance and Non-Abelian Gauge Field Theories

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Abstract

Following the line of reasoning of Utiyama in his classic paper [1], a brief and systematic study of the local non-abelian gauge principle in field theories is given.

1 Global Invariance

Let G be a non-Abelian Lie group with n-dimensional Lie algebra $\mathcal{L}g^n$

$$[X^a, X^b] = C^{ab}{}_c\, X^c,\ a = 1, 2, \cdots, n, \tag{1.1}$$

where $C^{ab}{}_c = -C^{ba}{}_c$ are the structure constants which satisfy the Jacobi identities

$$C^{ac}{}_d\, C^{bd}{}_e + C^{ba}{}_d\, C^{cd}{}_e + C^{cb}{}_d\, C^{ad}{}_e = 0.$$

Consider a theory $\hat{\mathcal{L}}(\varphi_i(x), \partial_\mu \varphi_i(x))$ with matter fields,$\varphi_i(x)$ carrying r-dimensional representation $\rho(X)$ of the Lie algebra

$$\delta\varphi_i(x) = \rho(X)\varphi_i(x) = \epsilon_a (T^a)_{ij}\, \varphi_j(x),\ i = 1, 2, \cdots, r\,, \tag{1.2}$$

where ϵ_a are the infinitesimal constant parameters of the Lie group G, and T^a, is a $r \times r$ matrix representation of $\mathcal{L}g^n$: $[T^a, T^b]_{ij} = C^{ab}{}_c T^c_{ij}$. From Eq(2) it follows that $\partial_\mu \varphi_i$ transforms covariantly (i.e., by the same representation matrices ρ of the field φ_i) under G,

$$\delta\left(\partial_\mu \varphi_i(x)\right) = \partial_\mu\left(\delta\varphi_i(x)\right) = \epsilon_a (T^a)_{ij}\, \partial_\mu \varphi_j(x). \tag{1.3}$$

The Lagrangian, therefore, transforms according to

$$\delta\hat{\mathcal{L}} = \frac{\partial \hat{\mathcal{L}}}{\partial \phi_i}\, \epsilon_a (T^a)_{ij}\, \varphi_j + \frac{\partial \hat{\mathcal{L}}}{\partial\left(\partial_\mu \varphi_i\right)}\, \epsilon_a (T^a)_{ij}\, \partial_\mu \varphi_j. \tag{1.4}$$

Thus, the necessary and sufficient conditions for $\hat{\mathcal{L}}$ to be invariant, $\delta\hat{\mathcal{L}} = 0$, under an arbitrary compact group of global internal transformations, are given by the following n identities,

$$G^a(\varphi) \doteq \frac{\partial \hat{\mathcal{L}}}{\partial \varphi_i}\, (T^a)_{ij}\, \varphi_j + \frac{\partial \hat{\mathcal{L}}}{\partial\left(\partial_\mu \varphi_i\right)}\, (T^a)_{ij}\, \partial_\mu \varphi_j = 0. \tag{1.5}$$

Using the definition of the Euler derivative

$$\frac{\delta \hat{\mathcal{L}}}{\delta \varphi_i} \doteq \frac{\partial \hat{\mathcal{L}}}{\partial \varphi_i} - \partial_\mu \left(\frac{\partial \hat{\mathcal{L}}}{\partial \left(\partial_\mu \varphi_i \right)} \right), \tag{1.6}$$

we can rewrite the condition of global invariance Eq(1.5) in the form

$$G^a(\varphi) = \frac{\delta \hat{\mathcal{L}}}{\delta \varphi_i} \left(T^a \right)_{ij} \varphi_j + \partial_\mu \left(\frac{\partial \hat{\mathcal{L}}}{\partial \left(\partial_\mu \varphi_i \right)} \left(T^a \right)_{ij} \varphi_j \right) = 0. \tag{1.7}$$

This identity is called the Noether identity. If the fields satisfy the Euler-Lagrange equations

$$\frac{\delta \hat{\mathcal{L}}}{\delta \varphi_i(x)} = 0, \tag{1.8}$$

then it follows, from Noether identity Eq(1.7), that the *current*, defined by

$$J^a_\mu(x) \doteq \frac{\partial \hat{\mathcal{L}}}{\partial \left(\partial^\mu \varphi_i \right)} \left(T^a \right)_{ij} \varphi_j(x), \tag{1.9}$$

is conserved.

2 The Noether Charge and its Properties

For convenience let us define the functional

$$Q[\sigma] = \int_\sigma d\sigma^\mu(x)\, J_\mu(x). \tag{2.1}$$

where σ denotes a space-like hypersurface in $\mathbb{R}^{(1,3)}$, and $d\sigma^\mu(x)$, is a n-vector differential at x. The functional derivative at some point x is defined by

$$\frac{\delta Q[\sigma]}{\delta \sigma(x)} \doteq \lim_{\omega(x) \to 0} \frac{Q[\bar{\sigma}] - Q[\sigma]}{\omega(x)}, \tag{2.2}$$

where $\omega(x)$ is the volume enclosed between $\bar{\sigma}$ and σ. Therefore, according to Gauss' theorem, we find

$$\frac{\delta Q[\sigma]}{\delta \sigma(x)} = \partial^\mu J_\mu. \tag{2.3}$$

Now, if $J_\mu(x)$ is conserved, then $\delta Q / \delta \sigma = 0$ and therefore $Q[\sigma]$ is independent of σ. This means that we are free to choose a particular $\sigma = x^0 = t =$ const. hyperplane to evaluate Q;

$$Q(t) = \int d^3x\, J_0(t, \vec{x}).$$

Clearly, this integral is time-independent iff the conserved vector field satisfies the boundary condition

$$|\vec{x}|^2 J_i(x) \to 0, \text{ as } |\vec{x}| \to \infty. \tag{2.4}$$

Indeed,

$$dQ/dt = \int d^3x\, \partial^0 J_0 = -\int d^3x\, \partial^i J_i = -\oint d\vec{S} \cdot \vec{J} = 0.$$

Next, we define, for some function $F(x)$, the functional

$$R_{\mu\nu}[\sigma] \doteq \int_\sigma d\sigma_\mu(x)\, \partial_\nu F(x). \tag{2.5}$$

Taking the functional derivative, we find

$$\frac{\delta R_{\mu\nu}[\sigma]}{\delta\sigma(x)} = \partial_\mu(\partial_\nu F) = \partial_\nu(\partial_\mu F) = \frac{\delta R_{\nu\mu}[\sigma]}{\delta\sigma(x)}.$$

Thus, $(R_{\mu\nu}[\sigma] - R_{\nu\mu}[\sigma])$ is independent of σ, i.e.,

$$\frac{\delta}{\delta\sigma(x)} \left(R_{\mu\nu}[\sigma] - R_{\nu\mu}[\sigma]\right) = 0. \tag{2.6}$$

Again, we are allowed to choose the hyperplane $\sigma = t =$ const. If we take, for example, $\mu = 0$, and $\nu = j$, we get

$$R_{0j} - R_{j0} = \int d^3x\, \partial_j F(x) - 0 = \int d^3x\, \partial_j F(x).$$

For $\mu = \nu = 0$, and for $\mu = i$, $\nu = j$, we have the following trivial results

$$R_{00} - R_{00} = \left(\int d^3x\, \partial_0 F(x) - \int d^3x\, \partial_0 F(x)\right) = 0,$$

$$R_{ij} - R_{ji} = 0 - 0 = 0.$$

Thus, if the function $F(x)$ satisfies the boundary condition

$$|\vec{x}|^2\, F(x) \to 0 \text{ as } |\vec{x}| \to \infty,$$

the following (very useful) identity [2] is satisfied for all values of μ and ν,

$$\int_\sigma d\sigma_\mu(x)\, \partial_\nu F(x) = \int_\sigma d\sigma_\nu(x)\, \partial_\mu F(x). \tag{2.7}$$

Now, the action of the translation generator P^μ on any local field $J_\mu(x)$ is given by

$$\delta^\mu J_\nu(x) = [iP^\mu, J_\nu(x)] = \partial^\mu J_\nu(x). \tag{2.8}$$

Integrating this equation we find by virtue of Eq(2.7), and assuming $|\vec{x}|^2 J_j \to 0$ as $|\vec{x}| \to \infty$, that

$$[iP^\mu, \int d\sigma^\nu\, J_\nu(x)] = \int d\sigma^\nu\, \partial^\mu J_\nu(x) = \int d\sigma^\mu\, \partial^\nu J_\nu.$$

Thus, for a conserved vector field, $\partial^\mu J_\mu = 0$, we find

$$[iP^\mu, Q] = 0.$$

That is, the integral over all space of the time component of a conserved vector, i.e., the Noether charge $Q \doteq \int d^3x\, J_0(x)$, is translationally invariant.

Under Lorentz transformation, $\exp(i\omega_{\mu\nu}\, M^{\mu\nu}/2)$, a vector field $J_\mu(x)$ transforms according to

$$\delta^{\mu\nu} J_\rho(x) = [iM^{\mu\nu},\, J_\rho(x)] = \partial^\nu\,(x^\mu\, J_\rho) - \partial^\mu\,(x^\nu\, J_\rho) + \delta^\mu_\rho\, J^\nu - \delta^\nu_\rho\, J^\mu. \tag{2.9}$$

Operating on this equation with $\int d\sigma^\rho$, and writing $Q = \int d\sigma^\rho\, J_\rho(x)$, we find

$$[iM^{\mu\nu}, Q] = \int d\sigma^\rho\, \partial^{[\nu}\left(x^{\mu]}\, J_\rho\right) + \int d\sigma^{[\mu}\, J^{\nu]}, \tag{2.10}$$

where

$$\partial^{[\nu}\left(x^{\mu]}\, J_\rho\right) \doteq \partial^\nu\,(x^\mu\, J_\rho) - \partial^\mu\,(x^\nu\, J_\rho)\,,$$

and

$$\int d\sigma^{[\mu}\, J^{\nu]} \doteq \int d\sigma^\mu\, J^\nu(x) - \int d\sigma^\nu\, J^\mu(x)\,.$$

If the vector field satisfies the boundary conditions

$$|\vec{x}|^2\, x^\mu\, J_\nu \to 0,\ \text{as } |\vec{x}| \to \infty,$$

we can use the identity Eq(2.7) to show that the following identity holds

$$\int d\sigma^\rho\, \partial^{[\nu}\left(x^{\mu]}\, J_\rho\right) = \int d\sigma^{[\nu}\, x^{\mu]}\, \partial^\rho J_\rho + \int d\sigma^{[\nu}\, J^{\mu]}. \tag{2.11}$$

Inserting this in Eq(2.10), leads to

$$[iM^{\mu\nu}, Q] = \int d\sigma^{[\nu}\, x^{\mu]}\, \partial^\rho J_\rho. \tag{2.12}$$

Thus the charge Q is Lorentz invariant, i.e., $[iM^{\mu\nu}, Q] = 0$, if the conserved Noether current $J_\mu(x)$ satisfies the above-mentioned boundary conditions. Thus, we have completed the proof the following theorem.

Theorem. *If the vector fields J^a_μ are conserved and well-behaved at spatial infinity, then the associated charges, defined by the integrals*

$$Q^a = \int d^3x\, J^a_0(x), \tag{2.13}$$

are Poincare-invariant and time-independent scalars.

Using the definition of the conjugate momentum

$$\pi_i(x) \doteq \frac{\partial \hat{\mathcal{L}}}{\partial\left(\partial^0 \varphi_i\right)},$$

the Noether charge becomes

$$Q^a = \int d^3x\, \pi_i(x)\, T^a_{ij}\, \varphi_j(x).$$

Owing to the equal-time commutation relations

$$\begin{gathered}[\varphi_i(t,\vec{x}), \pi_j(t,\vec{y}) = i\delta_{ij}\, \delta^3(\vec{x}-\vec{y}), \\ [\varphi_i(t,\vec{x}), \varphi_j(t,\vec{y})] = [\pi_i(t,\vec{x}), \pi_j(t,\vec{y})] = 0 \quad ,\end{gathered}$$

and the algebra of the matrices T^a, it is easy to see that the charges Q^a generate the correct transformation on the field variables,

$$\delta\varphi_i(x) = [\epsilon_a\, Q^a(t), \varphi_i(x)], \tag{2.14}$$

and form a representation of $\mathcal{L}g^n$,

$$[Q^a(t), Q^b(t)] = C^{ab}{}_c\, Q^c(t). \tag{2.15}$$

The remarkable fact about Eq(2.14) and Eq(2.15) is that they are true even if G is not a symmetry group, i.e., the charges Q^a satisfy the Lie algebra of G and generate the proper transformation on the fields regardless whether or not the currents $J^a_\mu(x)$ are conserved.

3 Local Gauge Invariance

Let us now assign an independent group element $g(\epsilon) \in G$ for each spacetime point x^μ, i.e., suppose now that the parameters of the group are arbitrary functions of the coordinate $\epsilon_a(x)$. The fields then transform according to

$$\delta\varphi_i(x) = (T^a)_{ij}\, \epsilon_a(x)\, \varphi_j(x). \tag{3.1}$$

The group of such transformations is called the *local* or *gauge* group. From Eq(3.1) it follows that the derivative of the field does not transform covariantly,

$$\delta\left(\partial_\mu\varphi_i(x)\right) = (T^a)_{ij}\, \epsilon_a(x)\, \partial_\mu\varphi_j(x) + (T^a)_{ij}\, \varphi_j(x)\, \partial_\mu\epsilon_a(x). \tag{3.2}$$

This equation implies that our *globally* invariant Lagrangian, $\hat{\mathcal{L}}(\varphi, \partial\varphi)$, is no longer invariant under the group, $G(x)$, of local transformations Eq(3.1). Indeed, we get for the change in the Lagrangian,

$$\delta\hat{\mathcal{L}} = \frac{\partial\hat{\mathcal{L}}}{\partial\varphi_i}\, \delta\varphi_i(x) + \frac{\partial\hat{\mathcal{L}}}{\partial\left(\partial_\mu\varphi_i\right)}\, \delta\left(\partial_\mu\varphi_i(x)\right) = G^a\, \epsilon_a(x) + \frac{\partial\hat{\mathcal{L}}}{\partial\left(\partial_\mu\varphi_i(x)\right)}\, (T^a)_{ij}\, \varphi_j(x)\, \partial_\mu\epsilon_a(x). \tag{3.3}$$

Therefore, the conditions of global invariance, Eq(1.5), imply that the variation of $\hat{\mathcal{L}}$ does not vanish for non-constant $\epsilon_a(x)$,

$$\delta \hat{\mathcal{L}}(x) = \frac{\partial \hat{\mathcal{L}}}{\partial\left(\partial_\mu \varphi_i\right)} \left(T^a\right)_{ij} \varphi_j \, \partial_\mu \epsilon_a(x). \tag{3.4}$$

This means that our matter field theory, $\hat{\mathcal{L}}(\varphi, \partial\varphi)$, is not invariant under the local group, $G(x)$, of transformations Eq(3.1). To obtain an invariant Lagrangian, it is necessary to enlarge the original (globally invariant) theory, $\hat{\mathcal{L}}(\varphi, \partial\varphi)$,by introducing a new field

$$A_J(x), \;\; J = 1, 2, \cdots, m, \tag{3.5}$$

in such a way that the right-hand side of Eq(3.4) can be cancelled with the contribution from this new field $A_J(x)$. The *compensating* fields thus introduced are called *gauge* fields.

Now suppose that the new Lagrangian $\bar{\mathcal{L}}(x)$ depends only on the fields $A_J(x)$ and not on their derivatives,

$$\bar{\mathcal{L}}(x) = \bar{\mathcal{L}}(\varphi_i, \partial_\mu \varphi_i, A_J), \tag{3.6}$$

$$\bar{\mathcal{L}}(\varphi_i, \partial_\mu \varphi_i, 0) \equiv \hat{\mathcal{L}}(\varphi_i, \partial_\mu \varphi_i),$$

and consider the following inhomogeneous infinitesimal transformations

$$\delta A_J(x) = F_J^{Ka} \, A_K(x) \epsilon_a(x) + G^a_{\mu J} \, \partial^\mu \epsilon_a(x), \tag{3.7}$$

where F and G are some unknown constants to be determined later. The second (inhomogeneous) term in Eq(3.7) has been introduced to cancel the right-hand side of Eq(3.4).

Now, the assumed invariance of $\bar{\mathcal{L}}(\varphi, \partial\varphi, A)$ under the local gauge group $G(x)$ reads

$$\delta \bar{\mathcal{L}} = \frac{\partial \bar{\mathcal{L}}}{\partial \varphi_i} \, \delta\varphi_i + \frac{\partial \bar{\mathcal{L}}}{\partial\left(\partial_\mu \varphi_i\right)} \, \delta\left(\partial_\mu \varphi_i\right) + \frac{\partial \bar{\mathcal{L}}}{\partial A_J} \, \delta A_J(x) = 0. \tag{3.8}$$

Inserting Eq(3.1), Eq(3.2) and Eq(3.7) in Eq(3.8) we get

$$\left(\frac{\partial \bar{\mathcal{L}}}{\partial \varphi_i} \, T^a_{ij} \, \varphi_j + \frac{\partial \bar{\mathcal{L}}}{\partial\left(\partial_\mu \varphi_i\right)} \, T^a_{ij} \, \partial_\mu \varphi_j + \frac{\partial \bar{\mathcal{L}}}{\partial A_J} \, F_J^{Ka} A_K(x) \right) \epsilon_a(x) + \left(\frac{\partial \bar{\mathcal{L}}}{\partial\left(\partial^\mu \varphi_i\right)} \, T^a_{ij} \, \varphi_j + \frac{\partial \bar{\mathcal{L}}}{\partial A_J} \, G^a_{\mu J} \right) \partial^\mu \epsilon_a(x) = 0. \tag{3.9}$$

Owing to the arbitrariness of the functions $\epsilon_a(x)$, the left-hand side of Eq(3.9) vanishes if and only if each coefficient of ϵ_a and $\partial_\mu \epsilon_a$ vanishes identically:

$$\frac{\partial \bar{\mathcal{L}}}{\partial \varphi_i} \, T^a_{ij} \, \varphi_j + \frac{\partial \bar{\mathcal{L}}}{\partial\left(\partial_\mu \varphi_i\right)} \, T^a_{ij} \, \partial_\mu \varphi_j + \frac{\partial \bar{\mathcal{L}}}{\partial A_J} \, F_J^{Ka} \, A_K(x) = 0, \tag{3.10}$$

$$\frac{\partial \bar{\mathcal{L}}}{\partial\left(\partial^\mu \varphi_i\right)} \, T^a_{ij} \, \varphi_j + \frac{\partial \bar{\mathcal{L}}}{\partial A_J} \, G^a_{\mu J} = 0. \tag{3.11}$$

What we are after next is to reveal the meaning of the index J carried by the field $A_J(x)$ and determine the explicit A_J-dependence of $\bar{\mathcal{L}}$. In order to determine that A_J-dependence uniquely, the number of equations Eq(3.11) has to equal the number (m) of the fields A_J

$$m = \dim\left(\mathbb{R}^{(1,3)}\right) \times \dim\left(\mathcal{L}g^n\right) = 4n.$$

Also, the $4n \times 4n$ matrix $G^a_{\mu J}$ must be non-singular with inverse matrix given by

$$\left(G^{-1}\right)^{\mu K}_{a} G^a_{\mu J} = \delta^K_J \,; \;\; \left(G^{-1}\right)^{\mu J}_{b} G^a_{\nu J} = \delta^\mu_\nu \, \delta^a_b. \tag{3.12}$$

We can use this to redefine the gauge field by

$$B^\mu_a(x) = \left(G^{-1}\right)^{\mu J}_{a} A_J(x) \,; \;\; A_J(x) = G^a_{\mu J} \, B^\mu_a(x). \tag{3.13}$$

In terms of the field B^μ_a the gauge transformation Eq(3.7) becomes

$$\delta B^\mu_c(x) = \left(\mathcal{C}^{ab}{}_c\right)^\mu_\nu B^\nu_b(x) \, \epsilon_a(x) + \partial^\mu \epsilon_c(x), \tag{3.14}$$

where

$$\left(\mathcal{C}^{ab}{}_c\right)^\mu_\nu \doteq \left(G^{-1}\right)^{\mu J}_{c} F^{Ka}_J \, G^b_{\nu K}. \tag{3.15}$$

For constant ϵ_a, therefore, the gauge fields transform according to

$$\delta B^\mu_c(x) = \left(\mathcal{C}^{ab}{}_c\right)^\mu_\nu \epsilon_a \, B^\nu_b(x). \tag{3.16}$$

Therefore, for non vanishing $(\mathcal{C}^{ab}{}_c)^\mu_\nu$, the gauge fields contribute to the Noether currents of the global symmetry group G and carry the corresponding charges. Thus, the gauge fields of the non-Abelian group G are self-interacting (charged) fields. Before continuing the field theoretical study, let us see what we can learn from the index structure of $B^\mu_a(x)$ and its transformation law Eq(3.14). Since B^μ_a carries single space-time index, therefore its field quanta are spin-1 vector bosons. The group index carried by $B^\mu_a(x)$, means that the gauge fields take values in the Lie algebra $\mathcal{L}g^n$, i.e., the theory contains as many gauge fields as there are generators. Thus, from Lie algebra point of view, the field B^μ_a transforms by the adjoint map

$$\mathrm{ad}(X)^b_c = C^{ab}{}_c \, \epsilon_a,$$

where ϵ_a are the *constant parameters* of the group. $C^{ab}{}_c = (X^a)^b{}_c$ are the matrix elements of the generators in the adjoint representation.The fact that the structure constants form a representation of $\mathcal{L}g^n$, can be easily seen from their Jacobi identities. Thus

$$\delta B^\mu_c(x) = \mathrm{ad}(X)^b_c \, B^\mu_b(x) = C^{ab}{}_c \, \epsilon_a \, B^\mu_b(x). \tag{3.17}$$

Comparing this with Eq(3.16) we conclude that the numbers $(\mathcal{C}^{ab}{}_c)^\mu_\nu$ are related to the structure constants of the group G by

$$\left(\mathcal{C}^{ab}{}_c\right)^\mu_\nu = C^{ab}{}_c \, \delta^\mu_\nu. \tag{3.18}$$

Using this result, the transformation law of the gauge field becomes

$$\delta B^\mu_c(x) = C^{ab}{}_c \, B^\mu_b(x) \, \epsilon_a(x) + \partial^\mu \epsilon_c(x). \tag{3.19}$$

This implies that the gauge bosons must be massless, i.e., Lagrangians with mass term $M^{ac} B_{\mu a} \, B^\mu_c$ are ruled out.

Using Eq(10) and Eq(28), it is easy to show that the combination

$$\mathcal{D}^\mu \varphi_i \doteq \partial^\mu \varphi_i - T^a_{ij} \, B^\mu_a \, \varphi_j, \tag{3.20}$$

transforms by the representation matrices $\rho(X^a) = T^a$ of the matter fields φ_i, i.e., it is a *covariant object,*

$$\delta\left(\mathcal{D}^\mu \varphi_i\right) = T^a_{ij}\, \epsilon_a(x)\, \mathcal{D}^\mu \varphi_j. \tag{3.21}$$

We will re-derive these results (Eq(3.18), Eq(3.19) and Eq(3.21)) by field theoretical considerations using Eq(3.10). So let us go back to field theory and rewrite Eq(3.11) in terms of the gauge field $B^\mu_a(x)$. Using the relations Eq(3.13) and Eq(3.12), Eq(3.11) becomes

$$\frac{\partial \bar{\mathcal{L}}}{\partial\left(\partial^\mu \varphi_i\right)}\, T^a_{ij}\, \varphi_j(x) + \frac{\partial \bar{\mathcal{L}}}{\partial B^\mu_a} = 0. \tag{3.22}$$

From this, Eq(3.22), we conclude that the gauge fields are contained in $\bar{\mathcal{L}}$ only through the combination, Eq(3.20),

$$\mathcal{D}^\mu \varphi_i(x) \doteq \partial^\mu \varphi_i - T^a_{ij}\, B^\mu_a\, \varphi_j,$$

which is called the *covariant derivative* (the name will be justified below). Thus, a gauge invariant Lagrangian should have the form

$$\bar{\mathcal{L}}(\varphi_i, \partial_\mu \varphi_i, A_J) = \mathcal{L}(\varphi_i, \mathcal{D}_\mu \varphi_i) \equiv \hat{\mathcal{L}}(\varphi_i, \mathcal{D}_\mu \varphi_i). \tag{3.23}$$

Therefore, the following relations must hold

$$\frac{\partial \bar{\mathcal{L}}}{\partial \varphi_i} = \frac{\partial \mathcal{L}}{\partial \varphi_i} - \frac{\partial \mathcal{L}}{\partial\left(\mathcal{D}^\mu \varphi_j\right)}\, T^a_{ji}\, B^\mu_a, \tag{3.24}$$

$$\frac{\partial \bar{\mathcal{L}}}{\partial\left(\partial^\mu \varphi_i\right)} = \frac{\partial \mathcal{L}}{\partial\left(\mathcal{D}^\mu \varphi_i\right)}, \tag{3.25}$$

$$\frac{\partial \bar{\mathcal{L}}}{\partial A_J} = -\frac{\partial \mathcal{L}}{\partial\left(\mathcal{D}^\mu \varphi_i\right)}\, T^a_{ij}\, \varphi_j\, (G^{-1})^{\mu J}_a. \tag{3.26}$$

Thus, a gauge invariant theory, $\mathcal{L}(\varphi, \mathcal{D}_\mu \varphi)$, can be obtained from a globally invariant matter field theory, $\hat{\mathcal{L}}(\varphi, \partial_\mu \varphi)$, by replacing the ordinary derivative $\partial_\mu \varphi$ with the *covariant derivative* $\mathcal{D}_\mu \varphi$. The second term in the covariant determines the coupling (interaction) between the matter fields $\varphi_i(x)$ and the Lie algebra valued gauge fields $B^\mu_a(x)$.Indeed, by expanding $\mathcal{L}(\varphi, \mathcal{D}^\mu \varphi)$ to first order in the coupling, we find

$$\mathcal{L}(\varphi_i, \mathcal{D}^\mu \varphi_i) = \hat{\mathcal{L}}(\varphi_i, \partial^\mu \varphi_i) - \frac{\partial \hat{\mathcal{L}}}{\partial\left(\partial^\mu \varphi_i\right)}\, T^a_{ij}\, \varphi_j\, B^\mu_a,$$

or, using Eq(9),

$$\mathcal{L}(\varphi_i, \mathcal{D}^\mu \varphi_i) = \hat{\mathcal{L}}(\varphi_i, \partial^\mu \varphi_i) - J^a_\mu(x)\, B^\mu_a(x). \tag{3.27}$$

Thus, the gauge fields couple to matter fields through the Noether currents of the global symmetry of the free matter field Lagrangian $\hat{\mathcal{L}}(\varphi, \partial\varphi)$. And $\mathcal{L}(\varphi, \mathcal{D}\varphi)$ is made up of the free matter Lagrangian $\hat{\mathcal{L}}(\varphi, \partial\varphi)$ and the interaction Lagrangian for the matter fields with the gauge fields $B^\mu_a(x)$,

$$\mathcal{L}_{\text{int}}(x) = -J^a_\mu(x)\, B^\mu_a(x). \tag{3.28}$$

Notice that the matter field's current can now be calculated from

$$\frac{\partial}{\partial B^\mu_a}\, \mathcal{L}(\varphi_i, \mathcal{D}^\nu \varphi_i) = -J^a_\mu(x) = -\frac{\partial \mathcal{L}}{\partial\left(\partial^\mu \varphi_i\right)}\, T^a_{ij}\, \varphi_j. \tag{3.29}$$

This means that the matter current J^a_μ is no longer conserved in usual sense. Instead, it satisfies a covariant conservation law which follows as a direct consequence of the invariance of $\mathcal{L}(\varphi, \mathcal{D}\varphi)$ under the *global* transformations

$$\delta\varphi_i(x) = T^a_{ij}\,\epsilon_a\,\varphi_j(x), \tag{3.30}$$
$$\delta B^\mu_c(x) = C^{ab}{}_c\,\epsilon_a\,B^\mu_b(x). \tag{3.31}$$

Indeed, the invariance of $\mathcal{L}$ implies

$$0 = \frac{\delta\mathcal{L}}{\delta\varphi_i}\,\delta\varphi_i + \epsilon_a\,\partial^\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial^\mu\varphi_i\right)}\,T^a_{ij}\,\varphi_j\right) + \epsilon_a\,C^{ab}{}_c\,B^\mu_b(x)\,\frac{\partial\mathcal{L}}{\partial B^\mu_c}.$$

Using Eq(3.29), we arrive at the Noether identity for the Lagrangian $\mathcal{L}(\varphi,\mathcal{D}\varphi)$:

$$\frac{\delta\mathcal{L}}{\delta\varphi_i}\,\delta\varphi_i + \epsilon_a\,\nabla^\mu J^a_\mu(x) = 0, \tag{3.32}$$

where ∇^μ,

$$\nabla^\mu J^a_\mu = \partial^\mu J^a_\mu - C^{ab}{}_c\,B^\mu_b\,J^c_\mu, \tag{3.33}$$

is the covariant derivative in the adjoint representation. Thus, when the field equations are satisfied, Noether identity implies that the matter current is covariantly conserved,

$$\nabla^\mu J^a_\mu = 0.$$

Next, we (as promised earlier) will use Eq(3.10) to show the covariant nature of the *covariant derivative,* Eq(3.21), and determine the unknown constants $(\mathcal{C}^{ab}{}_c)^\mu_\nu$ in the transformation law of the gauge field Eq(3.14). For that, let us examine Eq(3.10) term by term. From Eq(3.24), we can rewrite the first term in Eq(3.10)(after changing the dummy indices) as

$$\frac{\partial\bar{\mathcal{L}}}{\partial\varphi_i}T^a_{ik}\varphi_k = \frac{\partial\mathcal{L}}{\partial\varphi_i}T^a_{ik}\,\varphi_k - \frac{\partial\mathcal{L}}{\partial\left(\mathcal{D}^\mu\varphi_i\right)}B^\mu_b(x)\,\varphi_j\,T^b_{ik}\,T^a_{kj}. \tag{3.34}$$

From Eq(3.25), we find (after introducing the covariant derivative) for the second term of Eq(3.10),

$$\frac{\partial\bar{\mathcal{L}}}{\partial\left(\partial^\mu\varphi_i\right)}T^a_{ik}\,\partial^\mu\varphi_k = \frac{\partial\mathcal{L}}{\partial\left(\mathcal{D}^\mu\varphi_i\right)}T^a_{ik}\,\mathcal{D}^\mu\varphi_k + \frac{\partial\mathcal{L}}{\partial\left(\mathcal{D}^\mu\varphi_i\right)}B^\mu_b(x)\,\varphi_j\,T^a_{ik}\,T^b_{kj}. \tag{3.35}$$

Adding Eq(38) to Eq(39) and using the Lie algebra relation

$$T^a_{ik}\,T^b_{kj} - T^b_{ik}\,T^a_{kj} = [T^a,T^b]_{ij} = C^{ab}{}_c\,T^c_{ij}, \tag{3.36}$$

we find

$$\frac{\partial\bar{\mathcal{L}}}{\partial\varphi_i}T^a_{ij}\,\varphi_j + \frac{\partial\bar{\mathcal{L}}}{\partial\left(\partial^\mu\varphi_i\right)}T^a_{ij}\,\partial^\mu\varphi = \frac{\partial\mathcal{L}}{\partial\varphi_i}T^a_{ij}\,\varphi_j + \frac{\partial\mathcal{L}}{\partial\left(\mathcal{D}^\mu\varphi_i\right)}T^a_{ij}\,\mathcal{D}^\mu\varphi_j + \frac{\partial\mathcal{L}}{\partial\left(\mathcal{D}^\mu\varphi_i\right)}\varphi_j\,B^\nu_b\,\delta^\mu_\nu\,C^{ab}{}_c\,T^c_{ij}. \tag{3.37}$$

And finally we use Eq(3.26) to rewrite the third term in Eq(3.10) as

$$\frac{\partial\bar{\mathcal{L}}}{\partial A_J}\,F^{Ka}_J\,A_K = -\frac{\partial\mathcal{L}}{\partial\left(\mathcal{D}^\mu\varphi_i\right)}T^c_{ij}\,\varphi_j\,\left(G^{-1}\right)^{\mu J}_c\,F^{Ka}_J\,G^b_{\nu K}\,B^\nu_b = -\frac{\partial\mathcal{L}}{\partial\left(\mathcal{D}^\mu\varphi_i\right)}\,\varphi_j\,B^\nu_b\,\left(\mathcal{C}^{ab}{}_c\right)^\mu_\nu\,T^c_{ij}, \tag{3.38}$$

where the definition Eq(3.15) has been used. Substituting Eq(3.37) and Eq(3.38) in the identity Eq(3.10), we find

$$\frac{\partial\mathcal{L}}{\partial\varphi_i}T^a_{ij}\,\varphi_j + \frac{\partial\mathcal{L}}{\partial\left(\mathcal{D}^\mu\varphi_i\right)}T^a_{ij}\,\mathcal{D}^\mu\varphi_j = \frac{\partial\mathcal{L}}{\partial\left(\mathcal{D}^\mu\varphi_i\right)}T^c_{ij}\,\varphi_j\,B^\nu_b\,\left(\left(\mathcal{C}^{ab}{}_c\right)^\mu_\nu - \delta^\mu_\nu\,C^{ab}{}_c\right). \tag{3.39}$$

Thus, local gauge invariance demands that both sides of Eq(3.39) vanish identically. Indeed, the gauge invariance condition for $\mathcal{L}(\varphi_i, \mathcal{D}^\mu\varphi_i)$ reads

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi_i} T^a_{ij}\,\varphi_j\,\epsilon_a(x) + \frac{\partial\mathcal{L}}{\partial\left(\mathcal{D}^\mu\varphi_i\right)}\,\delta\left(\mathcal{D}^\mu\varphi_i\right) = 0. \tag{3.40}$$

By contracting Eq(3.39) with $\epsilon_a(x)$ and subtracting the result from Eq(3.40) we find

$$\delta\left(\mathcal{D}^\mu\varphi_i\right) - T^a_{ij}\,\epsilon_a(x)\left(\mathcal{D}^\mu\varphi_j\right) = T^c_{ij}\,\epsilon_a(x)\,\varphi_j\,B^\nu_b\left(\left(\mathcal{C}^{ab}{}_c\right)^\mu_\nu - \delta^\mu_\nu\,C^{ab}{}_c\right). \tag{3.41}$$

Thus, the local gauge invariance of $\mathcal{L}$ can be stated as

$$\left\{\left(\mathcal{C}^{ab}{}_c\right)^\mu_\nu = \delta^\mu_\nu\,C^{ab}{}_c\right\} \Leftrightarrow \left\{\delta B^\mu_c = C^{ab}{}_c\,\epsilon_a\,B^\mu_b + \partial^\mu\epsilon_c\right\} \Leftrightarrow \left\{\delta\left(\mathcal{D}^\mu\varphi_i\right) = T^a_{ij}\,\epsilon_a(x)\,\left(\mathcal{D}^\mu\varphi_j\right)\right\}. \tag{3.42}$$

We have already established the left to right implications (see Eq(3.18), Eq(3.19) and Eq(3.21)). The right to left implications can, just as easily, be proven:

$$\delta\left(\partial^\mu\varphi_i\right) - T^a_{ij}\,\left(B^\mu_a\,\delta\varphi_j + \delta B^\mu_a\,\varphi_j\right) = T^a_{ij}\,\epsilon_a\,\partial\varphi_j - T^a_{ij}\,T^b_{jk}\,B^\mu_b\,\varphi_k\,\epsilon_a.$$

Using Eq(10), Eq(3.2) and the algebra Eq(3.36), we find the transformation law of the gauge fields, Eq(3.19), which, when compared with Eq(3.14), leads to

$$\left(\mathcal{C}^{ab}{}_c\right)^\mu_\nu = \delta^\mu_\nu\,C^{ab}{}_c.$$

4 Completing The Dynamics

We have, up to this point, being treating the gauge field as external (non-propagating) field. indeed, we saw that the Lagrangian $\mathcal{L}(\varphi, \mathcal{D}^\mu\varphi)$ is made up of the free Lagrangian for the matter fields, $\hat{\mathcal{L}}(\varphi, \partial\varphi)$, and the interaction Lagrangian for the matter fields with the gauge fields, Eq(3.27) and Eq(3.28),

$$\mathcal{L}(\varphi, \mathcal{D}^\mu\varphi) = \hat{\mathcal{L}}(\varphi, \partial^\mu\varphi) + \mathcal{L}_{\text{int}}. \tag{4.1}$$

Let us now look for the possible, gauge invariant, Lagrangian for the gauge fields which depends on $B^\mu_a(x)$ as well as on their derivatives. Let it be denoted by $\mathcal{L}_0(B^\mu_a, \partial^\mu B^\nu_a)$. The invariance condition for $\mathcal{L}_0$ with respect to the gauge transformations Eq(3.19), reads

$$\delta\mathcal{L}_0 = \frac{\partial\mathcal{L}_0}{\partial B^\mu_a}\,\delta B^\mu_a + \frac{\partial\mathcal{L}_0}{\partial\left(\partial^\nu B^\mu_a\right)}\,\delta\left(\partial^\nu B^\mu_a\right) = 0. \tag{4.2}$$

Inserting Eq(3.19) in Eq(4.2) and using the arbitrariness of the functions $\epsilon_a(x)$, $\partial^\mu\epsilon_a(x)$, and $\partial^\mu\partial^\nu\epsilon_a(x)$, we find the following identities

$$\frac{\partial\mathcal{L}_0}{\partial B^\mu_c}\,C^{ab}{}_c\,B^\mu_b + \frac{\partial\mathcal{L}_0}{\partial\left(\partial^\nu B^\mu_e\right)}\,C^{ab}{}_e\,\partial^\nu B^\mu_b = 0, \tag{4.3}$$

$$\frac{\partial\mathcal{L}_0}{\partial B^\mu_c} + \frac{\partial\mathcal{L}_0}{\partial\left(\partial^\mu B^\nu_e\right)}\,C^{cd}{}_e\,B^\nu_d = 0, \tag{4.4}$$

$$\frac{\partial\mathcal{L}_0}{\partial\left(\partial^\nu B^\mu_a\right)} + \frac{\partial\mathcal{L}_0}{\partial\left(\partial^\mu B^\nu_a\right)} = 0. \tag{4.5}$$

The last identity follows from symmetrizing the product

$$\frac{\partial\mathcal{L}_0}{\partial\left(\partial^\nu B^\mu_a\right)}\,\partial^\nu\partial^\mu\epsilon_a = \frac{1}{2}\left\{\frac{\partial\mathcal{L}_0}{\partial\left(\partial^\mu B^\nu_a\right)} + \frac{\partial\mathcal{L}_0}{\partial\left(\partial^\nu B^\mu_a\right)}\right\}\,\partial^\mu\partial^\nu\epsilon_a(x),$$

and indicates that the derivative of the gauge field $B_a^\mu(x)$ can only enter the Lagrangian through the antisymmetric combination

$$B_a^{\mu\nu}(x) \doteq \partial^\mu B_a^\nu - \partial^\nu B_a^\mu. \tag{4.6}$$

Substituting Eq(4.4) into the first term of Eq(4.3) and using $C^{cd}{}_e = -C^{dc}{}_e$, we find

$$\frac{\partial \mathcal{L}_0}{\partial\left(\partial^\mu B_e^\nu\right)} \left\{C^{ab}{}_c\, C^{dc}{}_e\, B_b^\mu(x)\, B_d^\nu(x) + C^{ab}{}_e\, \partial^\mu B_b^\nu(x)\right\} = 0. \tag{4.7}$$

Anti-symmetrizing the indices (μ,ν), i.e., using Eq(4.5) or, which is the same thing, introducing $B_e^{\mu\nu}$ into Eq(4.7), we find

$$\frac{1}{2}\,\frac{\partial \mathcal{L}_0}{\partial B_e^{\mu\nu}(x)} \left\{C^{ab}{}_c\, C^{dc}{}_e\, B_b^\mu(x)\, B_d^\nu(x) - C^{ab}{}_c\, C^{dc}{}_e\, B_b^\nu(x)\, B_d^\mu(x) + C^{ab}{}_e\, B_b^{\mu\nu}(x)\right\} = 0. \tag{4.8}$$

Interchanging the dummy indices $(b \leftrightarrow d)$ in the second term and then using $C^{da}{}_c = -C^{ad}{}_c$, we find

$$\frac{1}{2}\,\frac{\partial \mathcal{L}_0}{\partial B_e^{\mu\nu}(x)} \left\{C^{ab}{}_e\, B_b^{\mu\nu}(x) + \left(C^{ab}{}_c\, C^{dc}{}_e + C^{da}{}_c\, C^{bc}{}_e\right)\, B_b^\mu(x)\, B_d^\nu(x)\right\} = 0. \tag{4.9}$$

Finally, using the Jacobi identity

$$C^{ab}{}_c\, C^{dc}{}_e + C^{da}{}_c\, C^{bc}{}_e = -C^{ac}{}_e\, C^{bd}{}_c, \tag{4.10}$$

we find

$$\frac{1}{2}\,\frac{\partial \mathcal{L}_0}{\partial B_e^{\mu\nu}}\, C^{ac}{}_e \left\{B_c^{\mu\nu}(x) - C^{bd}{}_c\, B_b^\mu(x)\, B_d^\nu(x)\right\} = 0, \tag{4.11}$$

or

$$\frac{1}{2}\,\frac{\partial \mathcal{L}_1(F_c^{\mu\nu})}{\partial F_e^{\mu\nu}(x)}\, C^{ac}{}_e\, F_c^{\mu\nu}(x) = 0, \tag{4.12}$$

where $F_a^{\mu\nu}(x)$ is the Lie algebra-valued tensor field

$$F_c^{\mu\nu}(x) = \partial^\mu B_c^\nu(x) - \partial^\nu B_c^\mu(x) - C^{bd}{}_c\, B_b^\mu(x)\, B_d^\nu(x), \tag{4.13}$$

and

$$\mathcal{L}_1(F_a^{\mu\nu}) \doteq \mathcal{L}_0(B_a^\mu, B_a^{\mu\nu}) \equiv \mathcal{L}_0(B_a^\mu, \partial^\nu B_a^\mu). \tag{4.14}$$

By contracting Eq(4.12) with $\epsilon_a(x)$, and comparing it with the invariance condition for $\mathcal{L}_1(F)$,

$$\delta\mathcal{L}_1 = \frac{\partial \mathcal{L}_1}{\partial F_c^{\mu\nu}}\, \delta F_c^{\mu\nu} = 0, \tag{4.15}$$

we deduce that $F_a^{\mu\nu}$ transforms in the adjoint representation of $\mathcal{L}g^n$

$$\delta F_a^{\mu\nu}(x) = C^{bc}{}_a\, \epsilon_b(x)\, F_c^{\mu\nu}(x). \tag{4.16}$$

Indeed, we can, as a consistency check, obtain the same transformation law for $F_a^{\mu\nu}$ by substituting Eq(3.19) in

$$\delta F_c^{\mu\nu} = \delta(\partial^\mu B_c^\nu) - \delta(\partial^\nu B_c^\mu) - C^{bd}{}_c\, B_b^\mu\, \delta B_d^\nu - C^{bd}{}_c\, (\delta B_b^\mu)\, B_d^\nu,$$

which follows from Eq(59), and using the Jacobi identity Eq(4.10). Thus, a locally invariant Lagrangian for the gauge fields is a function of the tensor $F_a^{\mu\nu}$ only and satisfies condition Eq(4.15). The choice of $\mathcal{L}_1(F)$ satisfying Eq(4.15) is not unique. The simplest Lorentz invariant and parity conserving Lagrangian, quadratic in $F_a^{\mu\nu}$, has the form

$$\mathcal{L}_1(F) = -\frac{1}{4}\, g^{ab}\, \eta_{\mu\rho}\, \eta_{\nu\sigma}\, F_a^{\mu\nu}(x)\, F_b^{\rho\sigma}(x), \tag{4.17}$$

with a constant non-singular matrix g^{ab} on $\mathcal{L}g^n$[1]. In order to have a real Lagrangian, the matrix g^{ab} must be real. Since

$$\frac{\partial \mathcal{L}_1}{\partial F_a^{\mu\nu}} = -\frac{1}{4}\left(g^{ab} + g^{ba}\right)\, \eta_{\mu\rho}\, \eta_{\nu\sigma}\, F_b^{\rho\sigma},$$

we may take g^{ab} to be symmetric matrix. Inserting Eq(4.17) in the gauge invariance condition, Eq(4.15), we find

$$g^{bc}\, C^{ad}{}_c = -g^{dc}\, C^{ab}{}_c. \tag{4.18}$$

Clearly, this is a condition on the allowed $\mathcal{L}g^n$. Indeed, if g^{ab} acts as a raising operator for the indices on the structure constants,

$$C^{abc} = g^{cd}\, C^{ab}{}_d, \tag{4.19}$$

then the condition Eq(4.18) shows that the structure constants are antisymmetric in all three indices a, b and d,

$$C^{abd} = -C^{adb}. \tag{4.20}$$

This implies that $\mathcal{L}g^n$ is a *compact* Lie algebra. With the aid of the Jacobi identity Eq(4.10), we can show that the Cartan metric, defined by

$$g^{ab} \doteq -\mathrm{Tr}\left\{\mathrm{ad}(X^a)\,\mathrm{ad}(X^b)\right\} = -C^{ac}{}_d\, C^{bd}{}_c, \tag{4.21}$$

satisfies the gauge group condition Eq(4.18). Then, a Lie algebra is said to be *compact* if the Cartan metric g^{ab} is *positive-definite:* since the finite-dimensional representations of compact Lie algebras are all Hermitian, $T^a = iX^a$, the Cartan metric $g^{ab} = \mathrm{Tr}(T^aT^b)$ is positive-definite as a bilinear form, because $g^{ab}\epsilon_a\epsilon_b = \mathrm{Tr}(T\cdot\epsilon)^2$ is positive for any real ϵ_a. Below, the compact nature of $\mathcal{L}g^n$ will be deduced on physical ground.

Now, if we rewrite the kinetic quadratic part,

$$-\frac{1}{4} g^{ab} \left(\partial^\mu B_a^\nu - \partial^\nu B_a^\mu\right)\left(\partial_\mu B_{\nu b} - \partial_\nu B_{\mu b}\right),$$

of the first term in Eq(4.17) as

$$+\frac{1}{2}\, g^{ab}\, \partial^\mu B_a^i\, \partial_\mu B_b^i - \frac{1}{2}\, g^{ab}\, \partial^\mu B_a^0\, \partial_\mu B_b^0 + \frac{1}{2}\, g^{ab}\, \partial^\mu B_a^\nu\, \partial_\nu B_{\mu b},$$

we see that the signature of the matrix g^{ab}(i.e., the signs of its eigenvalues) is related directly with the signs of (quantum) state-space metric of (transverse) gauge bosons B_a^μ. If g^{ab} has both positive and negative eigenvalues, it would be almost impossible to eliminate the contributions from negative norm states. Therefore, unless all modes in $B_a^\mu$are unphysical simultaneously, the Cartan metric must be of definite sign (in our case, positive definite) as a bilinear form. This implies that our gauge group G should be compact. Then, we can diagonalize and normalize the metric into the form

$$g^{ab} = \delta^{ab}.$$

With respect to this basis, we need not distinguish the upper and lower indices in the structure constants, and the gauge group restriction Eq(4.18) shows that C^{abc} is totally antisymmetric.

$$C^{ab}{}_c = -C^{ac}{}_b = C^{abc} \tag{4.22}$$

[1] To be precise, Eq(4.17) with non-singular matrix g^{ab} defines a correct Lagrangian only for the case with G being a *semi-simple* Lie group: Since g^{ab} is zero on any Abelian invariant subalgebra, Eq(4.17) does not reproduce the kinetic terms for Abelian components which appear in the case of non-semi-simple $\mathcal{L}g^n$.

Thus, the specific form, Eq(4.17), of $\mathcal{L}_1(F)$, which imposes the restriction Eq(4.18) on the allowed gauge group G, together with the existence of physical modes in B_a^μ restrict G to be compact. When the gauge group is compact, the invariant Lagrangian for the gauge fields is called Yang-Mills Lagrangian [3],

$$\mathcal{L}_{YM} = -\frac{1}{4}\, F^a_{\mu\nu}\, F^{\mu\nu}_a, \quad \text{where} \tag{4.23}$$

$$F^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu - C^{abc}\, B^b_\mu\, B^c_\nu, \tag{4.24}$$

is the Yang-Mills Field tensor. The Yang-Mills Lagrangian contains, beside the quadratic terms, the cubic and the quartic terms in the gauge fields B^a_μ, i.e., the Yang-Mills field is *self-interacting.* The *total Lagrangian* of the system of the matter fields $\varphi_i(x)$ and of the gauge fields $B^\mu_a(x)$ will be given by the sum of the original matter field Lagrangian $\hat{\mathcal{L}}(\varphi, \partial\varphi)$, the interaction Lagrangian between the matter and the gauge fields Eq (3.28) and the Yang-Mills Lagrangian,

$$\mathcal{L}(\varphi_i, \partial\varphi_i, B^a_\mu, \partial B^a_\mu) = \hat{\mathcal{L}}(\varphi_i, \partial\varphi_i) - J^a_\mu(x)\, B^\mu_a(x) - \frac{1}{4}\, F^a_{\mu\nu}\, F^{\mu\nu}_a \equiv \hat{\mathcal{L}}(\varphi_i, \mathcal{D}_\mu\varphi_i) - \frac{1}{4}\, F^a_{\mu\nu}\, F^{a\mu\nu}, \tag{4.25}$$

with $J^a_\mu(x)$, given by Eq(1.9), being the matter field's current of the *global* symmetry of $\hat{\mathcal{L}}(\varphi, \partial\varphi)$.

5 Conserved Currents; First and Second Noether Theorems

In this section, we will state and prove the two theorems of Noether in the case of the theory $\mathcal{L}(\varphi, \partial\varphi, B^a_\mu, \partial B^a_\mu)$.

Theorem 1. *The Lagrangian density $\mathcal{L}(\varphi, \partial\varphi, B^\mu_a, \partial B^\mu_a)$ is invariant under the following infinitesimal transformations*

$$\delta\varphi_i(x) \doteq \epsilon_a \delta^a \varphi_i(x) = T^a_{ij}\, \epsilon_a\, \varphi_j(x), \tag{5.1}$$

$$\delta B^\mu_c(x) \doteq \epsilon_a \delta^a B^\mu_c(x) = C^{ab}{}_c\, \epsilon_a\, B^\mu_b(x), \tag{5.2}$$

with arbitrary constant parameters ϵ_a, if and only if the following (Noether) identity holds

$$\frac{\delta\mathcal{L}}{\delta\Phi_A}\, \delta^a\Phi_A + \partial^\mu \mathcal{J}^a_\mu(x) = 0, \tag{5.3}$$

where

$$\mathcal{J}^a_\mu(x) \doteq \frac{\partial\mathcal{L}}{\partial\,(\partial^\mu\varphi_i)}\, T^a_{ij}\, \varphi_j(x) + \frac{\partial\mathcal{L}}{\partial\,(\partial^\mu B^\nu_c)}\, C^{ab}{}_c B^\nu_b(x), \tag{5.4}$$

$$\frac{\delta\mathcal{L}}{\delta\Phi_A}\, \delta^a\Phi_A \doteq \frac{\delta\mathcal{L}}{\delta\varphi_i}\, \delta^a\varphi_i + \frac{\delta\mathcal{L}}{\delta B^\mu_c}\, \delta^a B^\mu_c. \tag{5.5}$$

Proof. The infinitesimal global transformations above induce the following change in the Lagrangian

$$\epsilon_a\, \delta^a\mathcal{L} = \epsilon_a\, \frac{\partial\mathcal{L}}{\partial\varphi_i}\, \delta^a\varphi_i + \epsilon_a\, \frac{\partial\mathcal{L}}{\partial\,(\partial^\mu\varphi_i)}\, \partial^\mu(\delta^a\varphi_i) + \epsilon_a\, \frac{\partial\mathcal{L}}{\partial B^\nu_c}\, \delta^a B^\nu_c + \epsilon_a\, \frac{\partial\mathcal{L}}{\partial\,(\partial^\mu B^\nu_c)}\, \partial^\mu(\delta^a B^\nu_c).$$

After introducing the Euler derivatives for the fields $\Phi_A = \{\varphi_i, B^\mu_c\}$, and owing to the arbitrariness of the constant parameters ϵ_a, the above equation becomes

$$\delta^a\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\Phi_A}\, \delta^a\Phi_A + \partial^\mu \left(\frac{\partial\mathcal{L}}{\partial\,(\partial^\mu\varphi_i)}\, T^a_{ij}\, \varphi_j(x) + \frac{\partial\mathcal{L}}{\partial\,(\partial^\mu B^\nu_c)}\, C^{ab}{}_c\, B^\nu_b(x) \right). \tag{5.6}$$

Thus $\delta^a\mathcal{L} = 0$ if and only if the right-hand side of Eq(5.6) vanishes identically (i.e., irrespective of whether or not Φ^A 's are solutions of the field equations). $\square$

When the field equations are satisfied,

$$\frac{\delta \mathcal{L}}{\delta \Phi_A} = 0,$$

Noether identity Eq(5.3) shows that the *global symmetry* current $\mathcal{J}^a_\mu$ is conserved. Using the Yang-Mills Lagrangian Eq(4.23) and the definition Eq(1.9) of the matter current, we can rewrite the current $\mathcal{J}^a_\mu$ of the global symmetry of the total Lagrangian Eq(4.25) as

$$\mathcal{J}^a_\mu(x) = J^a_\mu(x) + C^{ab}{}_c\, B^\nu_b(x)\, F^c_{\nu\mu}(x). \tag{5.7}$$

The second term represents the Gauge fields contribution to the conserved current. Thus, for non-abelian symmetry the charges carried by the gauge bosons are given by

$$q^a = C^{ab}{}_c \int d^3x\, F^c_{j0}(x)\, B^j_b(x). \tag{5.8}$$

Notice that, while the gauge-dependent current$\mathcal{J}^a_\mu$ is conserved, the gauge-invariant (matter) current J^a_μ is not. We will see that the matter field current satisfies covariant conservation law.

Theorem 2. *The Lagrangian $\mathcal{L}(\varphi_i, \partial^\mu \varphi_i, B^\mu_a, \partial^\nu B^\mu_a)$ is invariant under the infinitesimal transformations*

$$\delta\varphi_i(x) = T^a_{ij}\, \epsilon_a(x)\, \varphi_j(x),$$

$$\delta B^\nu_c(x) = C^{ab}{}_c\, \epsilon_a(x)\, B^\nu_b(x) + \partial^\nu \epsilon_c(x),$$

with arbitrary, twice differentiable, spacetime-dependent functions $\epsilon_a(x)$, if and only if the following relations hold identically:

$$\frac{\delta \mathcal{L}}{\delta \Phi_A}\, \delta^a \Phi_A - \partial^\nu \left(\frac{\delta \mathcal{L}}{\delta B^\nu_a} \right) = 0, \tag{5.9}$$

$$\partial^\nu \mathbb{F}^c_{\nu\mu}(x) + \mathbb{J}^c_\mu(x) = 0, \tag{5.10}$$

$$\mathbb{F}^a_{\mu\nu}(x) = -\mathbb{F}^a_{\nu\mu}(x). \tag{5.11}$$

With $\mathbb{J}^a_\mu$ and $\mathbb{F}^a_{\mu\nu}$ are defined by

$$\mathbb{J}^c_\mu(x) = \frac{\delta \mathcal{L}}{\delta B^\mu_c} + \mathcal{J}^c_\mu(x); \tag{5.12}$$

$$\mathcal{J}^a_\mu(x) = \frac{\partial \mathcal{L}}{\partial\left(\partial^\mu \varphi_i\right)}\, T^a_{ij}\, \varphi_j + \frac{\partial \mathcal{L}}{\partial\left(\partial^\mu B^\nu_c\right)}\, C^{ab}{}_c\, B^\nu_b,$$

$$\mathbb{F}^a_{\mu\nu}(x) = \frac{\partial \mathcal{L}}{\partial\left(\partial^\nu B^\mu_a\right)}. \tag{5.13}$$

Proof. This does not require anything other that introducing the Euler derivatives for the fields and some reshuffling of the terms:

$$\delta\mathcal{L} = \left\{ \frac{\delta \mathcal{L}}{\delta \Phi_A}\, \delta^a \Phi_A - \partial^\nu \left(\frac{\delta \mathcal{L}}{\delta B^\nu_a} \right) + \partial^\nu \mathbb{J}^a_\nu \right\} \epsilon_a + \left\{ \mathbb{J}^c_\mu + \partial^\nu \mathbb{F}^c_{\nu\mu} \right\} \partial^\mu \epsilon_c + \frac{1}{2} \left\{ \mathbb{F}^c_{\mu\nu} + \mathbb{F}^c_{\nu\mu} \right\} \partial^\mu \partial^\nu \epsilon_c.$$

Owing to the arbitrariness of functions $\epsilon_a(x)$, the change in $\mathcal{L}$ vanishes if and only if each coefficient of ϵ_a, $\partial^\mu \epsilon_a$ and $\partial^\mu \partial^\nu \epsilon_a$ vanishes identically. Thus the invariance of $\mathcal{L}$ is equivalent to the identities Eq(5.9), Eq(5.10) and Eq(5.11). □

When the matter fields satisfy the field equations $\delta\mathcal{L}/\delta\varphi_i = 0$, the identity Eq(5.9) becomes

$$\frac{\delta \mathcal{L}}{\delta B^\mu_c}\, \delta^a B^\mu_c - \partial^\nu \left(\frac{\delta \mathcal{L}}{\delta B^\nu_a} \right) = 0. \tag{5.14}$$

This means that the $4n$ quantities $(\delta\mathcal{L}/\delta B_c^\mu)$ are interrelated by n equations, and hence, that the number of independent quantities among $(\delta\mathcal{L}/\delta B_c^\mu)$ is $3n$ in general. Owing to this *constraint,* the theory cannot determined time evolution of the gauge field uniquely. The notion of quantum fields necessarily involves field operators or their Green's functions with some specified spacetime dependence determined by field equations. To quantize the gauge field, therefore, it is necessary to break the local gauge invariance by some *gauge fixing* condition. Of course, we encounter the same thing in the abelian gauge theory QED, however, what is new here is that the Lagrangian, Eq(4.25), together with the gauge fixing term cannot define a meaningful quantum theory of Yang-Mills fields with a unitary S-matrix. This is just an inevitable consequence of the non-linear self-interaction of gauge fields due to the non-abelian nature of the theory: Unlike the abelian cases, the contributions from the unphysical (longitudinal and scalar) modes to intermediate states do not exactly cancel out owing to the self-coupling of B_a^μ. Feynmann [4] and later DeWitt [5] found, in the perturbation theory, that this violation of unitarity can be restated as the missing contributions of a pair of massless *scalar fermions* to closed loops in the Feynmann diagrams. Subsequently, a clear explanation for the appearance of these fermions with strange statistics was given by Faddeev and Popov [6] from the viewpoint of path-integral formalism, and since then, these "particles" have come to be called *Faddeev-Popov ghosts.* Understanding the origin of these ghost fields in the operator formalism will be the subject of next set of notes.

From Eq(5.10) and Eq(5.11) it follows that the "current" $\mathbb{J}_\mu^a$ is conserved. However, this is not a new current. For B_a^μ satisfying the field equations $\delta\mathcal{L}/\delta B_a^\mu = 0$, the "current" $\mathbb{J}_\mu^a$defined by Eq(5.12) becomes identical to the conserved current $\mathcal{J}_\mu^a$ associated with the global transformation of the first Noether theorem . Thus, there is no new current associated with local gauge invariance. In the case of the Lagrangian Eq(4.25), $\mathbb{F}_{\mu\nu}^a$ is nothing but the field strength $F_{\mu\nu}^a$. Thus, using Eq(5.7) we can rewrite Eq(5.10) as

$$\partial^\nu F_{\nu\mu}^a = -J_\mu^a(x) - C^{abc}\, B_b^\nu\, F_{\nu\mu}^c, \tag{5.15}$$

or, in terms of the covariant derivative ∇^ν,

$$\nabla^\nu F_{\nu\mu}^a = -J_\mu^a(x). \tag{5.16}$$

From this it follows that the matter current satisfies a covariant conservation law:

$$\nabla^\mu \nabla^\nu F_{\nu\mu}^a = -\frac{1}{2}\,[\nabla^\mu, \nabla^\nu]\, F_{\mu\nu}^a = -\nabla^\mu J_\mu^a,$$

or, because of the total antisymmetry of the structure constants,

$$\nabla^\mu J_\mu^a = \frac{1}{2}\, C^{abc}\, F^{b\mu\nu}\, F_{\mu\nu}^c = 0. \tag{5.17}$$

6 Conclusions

The equirement that the Lagrangian be invariant under arbitrary local group of internal symmetry Eq(3.1) forced us to introduce new massless vector bosons transforming in the adjoint representation of the gauge group Eq(3.19). With no extra input other than gauge invariance, we were able to determine the form of interaction between the gauge bosons and the matter fields Eq(3.28), and show how the locally invariant Lagrangian can be deduced from the globally invariant Lagrangian Eq(3.23). We have also seen that the form of the Lagrangian for the gauge fields Eq(4.17) together with the existence of physical modes in $B_a^\mu(x)$ restricts the allowed gauge group to be compact. And finally, we deduced, from the results of Noether second theorem Eq(5.14), that gauge field theories are constraint systems.

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