GEOMETRIC ALGEBRA OF SPINS

February 13, 2022

ABSTRACT

This paper presents a brief introduction to geometric algebra and demonstrates the application to quantum mechanics with spin wave vectors. The approach here differs from previous approaches in that it uses an alternative geometric algebra expression for the wave vector, such that the probability of measurement simplifies to an inner product of state vectors. This has attractive properties for the algebra of transformations in quantum mechanics. Finally, it is shown how the expressions can be related to spinors.

1 Introduction

Geometric algebra - also known as Clifford algebra - provides a powerful mathematical structure for applications in the physics for rigid body motion, electromagnetism and more. In this paper the application to the algebra of spin quantum mechanics is shown. The approach differs from previous approaches in that special emphasis is put on the equation for the probability of measurement, which is usually given by the Born rule. Starting from a geometric algebra expression for the wave vector, the corresponding expressions for probability of measurement, spatial rotations and qubit gates are derived.

2 Introduction to geometric algebra

2.1 Algebra of vector multiplication

Vector algebra defines axioms for vector addition and multiplication by a real scalar number. Geometric algebra adds an additional multiplication between vectors. It is called geometric product and written as ab for two vectors a and b. This product is associative $(ab)c = a(bc)$ (unlike the vector dot product), but in general not commutative $ab \neq ba$ (like matrix multiplication). Scalars still commute with this new multiplication. This product is also distributive over addition $a(b + c) = ab + ac$, $(a + b)c = ac + bc$ and hence follows the same rules as matrix multiplication.

In order to create a fruitful structure, geometric algebra adds an axiom that vectors square to a real number

$$aa \in \mathbb{R}$$

For a vector space it is possible to find a set of orthonormal basis vectors

$$e_1, e_2, \ldots, e_n$$

where the maximum number n of these vectors is the dimension of the vector space. In geometric algebra these orthonormal basis vectors obey

$$\begin{aligned} e_i e_i &= 1 \\ e_i e_j &= -e_j e_i \qquad (i \neq j) \end{aligned} \tag{1}$$

This is how orthonormality is defined in geometric algebra, since the equivalent of the inner product can be defined as $a \cdot b = \frac{1}{2}(ab + ba)$. Hence, basis vectors square to 1 and different basis vectors anticommute. Technically, it is possible to have a geometric algebra where for some normal vectors $e_i^2 = -1$ or $e_i^2 = 0$, but here these different signature algebras are not used. These are essentially all the rules that are needed to verify and understand all of the following. You could essentially *treat the orthonormal basis vectors as matrices* which follow equations (1).

2.2 Multivectors as general elements

By successively multiplying basis vectors we can generate the whole algebra. The general element of geometric algebra is called a *multivector* and has the basis

$$(1, e_i, \ldots, e_i e_j, \ldots, e_i e_j e_k, \ldots)$$

since these products cannot be reduced any further. Usually the basis vector products are reordered by index and one writes $e_{ij\cdots} = e_i e_j \cdots$ as an abbreviation. For example the most general multivector for the three-dimensional basis e_1, e_2, e_3 is

$$A = \alpha_0 + \alpha_1 e_1 + \alpha_1 e_1 + \alpha_1 e_1 + \alpha_{12} e_{12} + \alpha_{13} e_{13} + \alpha_{23} e_{23} + \alpha_{123} e_{123}$$

which has 8 components and the coefficients α are real numbers. A multivector from n basis vectors has up to 2^n components. All other expressions can be reduced to this form by the above rules. A multivector consisting of a sum of products of exactly k basis vectors is said to have grade k. Scalars are grade-0 elements; vectors having only terms with $e_1, e_2, \ldots, e_n$ are grade-1 elements.

Multivectors are usually written with capital letters A, whereas grade-1 vectors are written with small letters a. *Multivector multiplication* inherits the same *associativity* and *distributivity* rules as vectors.

2.3 Additional operations

A useful operation is the extraction of the scalar part (grade 0) which is written as

$$\langle A \rangle$$

In our three dimensional example we have $\langle A \rangle = \alpha_0$. If a subscript is used as in $\langle A \rangle_k$, it means to extract the part with grade k.

Another useful operation is the reverse of a multivector which is written as

$$A^\dagger$$

It reverses the order of a multiplication, e.g. $(abc)^\dagger = cba$ for vectors and $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$ for multivectors. You can verify that for multivectors of grade $4j + \{2, 3\}$ the sign changes with the reverse operation (e.g. $(e_1 e_2)^\dagger = e_2 e_1 = -e_1 e_2$). For sums $(A + B)^\dagger = A^\dagger + B^\dagger$.

For many - but not all - multivectors it is possible to find an inverse multivector $AA^{-1} = 1$. In particular for grade-1 vectors or products of vectors $A = abc \cdots$

$$a^{-1} = \frac{1}{a^2} a \qquad\qquad A^{-1} = \frac{1}{AA^\dagger} A^\dagger$$

where is first factor turns out to be a scalar. Some multivectors like $e_1 e_2 + e_3 e_4$ do not have an inverse.

You can also find that expressions of the form $\alpha + J\beta$ with scalars α, β and the bivector $J = e_0 e_1$ follow the same rules as complex numbers. For example $J^2 = e_0 e_1 e_0 e_1 = -e_0 e_1 e_1 e_0 = -1$.

For a more formal introduction to geometric algebra and it's applications to physics you can refer to [Doran and Lasenby, 2007].

3 Wave vector in geometric algebra

The usual complex n-dimensional spin wave vector ψ can be written as the geometric algebra expression

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix} \quad \leftrightarrow \quad \Psi = \sum_{k=1}^{n} \frac{1}{\sqrt{2}} \left(e_k + J\, f_k\right) \left(\Re\psi_k + J\, \Im\psi_k\right) \tag{2}$$

where $\psi_k = \Re\psi_k + i\, \Im\psi_k$ is a component of the spin wave vector with real and imaginary components, e_i, f_i are a set of distinct orthonormal vectors in a $2n + 2$ dimensional vector space with $e_i^2 = +1, f_i^2 = +1$, and $J = e_0 f_0$ is a bivector which helps reproducing the complex multiplication. All coefficients are real numbers and no complex numbers are required. J commutes with all basis vectors $e_{i\geq 1}, f_{i\geq 1}$ and itself. The symbol $\leftrightarrow$ will be used to denote a translation of the conventional approach to geometric algebra.

The motivation to use this particular expression is that the probability to measure a state $\psi^{(A)}$ in another state $\psi^{(B)}$ which is given by the Born rule, can be calculated with geometric algebra from a simple expression

$$P(A \to B) = \left|\left\langle \psi^{(A)} | \psi^{(B)} \right\rangle\right|^2 \qquad \leftrightarrow \qquad P(A \to B) = \left\langle \Psi^{(A)} \Psi^{(A)\dagger} \Psi^{(B)} \Psi^{(B)\dagger} \right\rangle - 1 \tag{3}$$

The proof for this equation is presented in the appendix A.

Other treatments of geometric algebra for spin wave vectors usually use an expression equivalent to $\Psi = \sum_k e_k(\Re\psi_k + I\,\Im\psi_k)$ with a pseudoscalar $I = \prod_i e_i$. While all approaches reproduce quantum mechanics correctly, only for an expression as (2) the probability of measurement simplifies to (3).

4 State vector and measurement of probabilities

The expression in equation (3) can be further simplified by introducing the state vector

$$\Omega = J(\Psi\Psi^\dagger - 1) \tag{4}$$

This definition is chosen to get an insightful expression for a single spin-1/2 state vector. The probabilities become

$$P(A \to B) = \left\langle \Omega^{(A)} \Omega^{(B)\dagger} \right\rangle \tag{5}$$

since $\langle \Psi\Psi^\dagger \rangle = 1$ for normalized wave functions (see appendix B). This the same as the dot product between real vectors as you can verify that for general multivectors with distinct basis vector products $E_i = e_{ab\cdots}$

$$\left\langle \sum_i \alpha_i E_i \sum_j \beta_j E_j^\dagger \right\rangle = \sum_{ij} \alpha_i \beta_j \left\langle E_i E_j^\dagger \right\rangle = \sum_{ij} \alpha_i \beta_j \delta_{ij} = \sum_i \alpha_i \beta_i$$

Only products $\alpha_i\beta_j$ from the same basis multivectors $i = j$ contribute to the scalar result. The coefficients α_i and β_j could be put into a plain column vector and the dot product could be applied.

The state vector can be derived to be the expression

$$\begin{aligned} \Omega &= J(\Psi\Psi^\dagger - 1) \\ &= \sum_i e_i f_i\, |\psi_i|^2 + \sum_{i<j} (e_i f_j + e_j f_i)\, \Re(\psi_i \psi_j^*) - \sum_{i<j} (e_i e_j + f_i f_j)\, \Im(\psi_i \psi_j^*) \end{aligned} \tag{6}$$

when written in terms of the original spin wave vector components. This expression does not depend on J anymore. The derivation can be found in appendix B. While this representation could be directly taken from the density matrix, the development shown here provides the algebra to do that without artificially plucking the components $|\psi_i|^2, \Re(\psi_i\psi_j^*), \Im(\psi_i\psi_j^*)$ from the density matrix.

5 Single spin state vector, Bloch sphere and rotations in space

5.1 State vector

A single spin-1/2 particle can be represented by a 2-dimensional complex wave vector and be translated to geometric algebra with equation (2) as

$$\psi = \begin{pmatrix} a + bi \\ c + di \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{pmatrix} \qquad \leftrightarrow \qquad \Psi = \frac{1}{\sqrt{2}}\left(e_1(a + bJ) + f_1(aJ - b) + e_2(c + dJ) + f_2(cJ - d)\right)$$

where θ, ϕ are Euler angles.

However, it is more instructive to look at the state vector from equation (6)

$$\begin{aligned} \Omega &= J(\Psi\Psi^\dagger - 1) \\ &= \frac{1}{2}(e_1 f_1 + e_2 f_2) + \frac{1}{2}(e_2 f_2 - e_1 f_1)\cos\theta + \frac{1}{2}(e_1 f_2 + e_2 f_1)\sin\theta\cos\phi + \frac{1}{2}(e_1 e_2 + f_1 f_2)\sin\theta\sin\phi \end{aligned} \tag{7}$$

which is derived in the appendix C.

5.2 Spatial basis bivectors

We can define new bivectors and write this as

$$\Omega = T + Xx + Yy + Zz \tag{8}$$

$$X = \frac{1}{2}\left(e_1 f_2 + e_2 f_1\right) \qquad Y = \frac{1}{2}\left(e_1 e_2 + f_1 f_2\right) \qquad Z = \frac{1}{2}(e_2 f_2 - e_1 f_1) \qquad T = \frac{1}{2}(e_1 f_1 + e_2 f_2)$$
$$x = \sin\theta\cos\phi \qquad y = \sin\theta\sin\phi \qquad z = \cos\theta$$

This is the first time that we can relate a combination of the orthonormal vectors to coordinates in space, if we accept that space coordinates can be represented as bivectors. Equation (8) effectively describes a Bloch sphere.

The probability to measure on state in another state is the inner product $P = \langle \Omega^{(A)} \Omega^{(B)} \rangle$.

5.3 Spatial rotations

Spatial rotations around an axis given by the normed vector $\hat{n} = (r_x, r_y, r_z)$ and an angle α can be accomplished by

$$\Omega' = R\,\Omega\,R^\dagger \tag{9}$$

with the multivector

$$\begin{aligned} R &= \exp\left(\frac{\alpha}{2}(T + Xr_x + Yr_y + Zr_z)\right) \\ &= \cos\frac{\alpha}{2} + (T + Xr_x + Yr_y + Zr_z)\sin\frac{\alpha}{2} \end{aligned} \tag{10}$$

where R is known as a rotor due to $RR^\dagger = 1$. It is helpful to confirm that $(T + Xr_x + Yr_y + Zr_z)^2 = -1$. The algebraic validation can be aided by the multiplication formulas in appendix D. The real coefficients x, y, z transform as you would expect from rotations in three dimensions.

Due to $\Omega = J(\Psi\Psi^\dagger - 1)$, $RR^\dagger = 1$ and J commuting with the rotor, the same rotor can be applied one-sided to the wave vector

$$\Psi' = R\Psi \tag{11}$$

since

$$\Omega' = J(\Psi'\Psi'^\dagger - 1) = J(R\Psi\Psi^\dagger R^\dagger - 1) = RJ(\Psi\Psi^\dagger - 1)R^\dagger = R\Omega R^\dagger$$

But there is nothing special to the one-sided application, as a two-sided application $\Psi' = R\Psi R^\dagger$ would work as well.

We can use equation (10) and (11) to transform the wave function by any rotation.

References

Chris Doran and Anthony Lasenby. *Geometric Algebra for Physicists*. Cambridge University Press, 2007.

A Derivation of probability of measurement

We want to reproduce the Born rule

$$P(A \to B) = \left|\left\langle \psi^{(A)} | \psi^{(B)} \right\rangle\right|^2$$

With the geometric algebra wave vector

$$\Psi = \sum_i \Lambda_i \psi_i \qquad \Lambda_i = \frac{1}{\sqrt{2}}(e_i + Jf_i)$$

and the task to reproduce the probability of measuring state $\Psi^{(A)}$ in another state $\Psi^{(B)}$ we can calculate the expression

$$\left\langle \Psi^{(A)}\Psi^{(A)\dagger}\Psi^{(B)}\Psi^{(B)\dagger} \right\rangle = \left\langle \sum_{ijkl} \Lambda_i \Lambda_j^\dagger \Lambda_k \Lambda_l^\dagger \; \psi_i^{(A)} \psi_j^{(A)\dagger} \psi_k^{(B)} \psi_l^{(B)\dagger} \right\rangle$$

A contribution to the scalar result can only come from $\Lambda_i \Lambda_j^\dagger \Lambda_k \Lambda_l^\dagger$ terms which have a scalar part, or a scalar part when multiplied by J. Since, we are only interested in terms which contribute to the scalar result, we can use the equation

$$\left\langle \Lambda_i \Lambda_j^\dagger \Lambda_k \Lambda_l^\dagger \right\rangle = \delta_{il}\delta_{jk} + \delta_{ij}\delta_{kl} \tag{12}$$

Therefore

$$\begin{aligned}
\left\langle \Psi^{(A)}\Psi^{(A)\dagger}\Psi^{(B)}\Psi^{(B)\dagger} \right\rangle &= \left\langle \sum_{ijkl} (\delta_{il}\delta_{jk} + \delta_{ij}\delta_{kl})\ \psi_i^{(A)}\psi_j^{(A)\dagger}\psi_k^{(B)}\psi_l^{(B)\dagger} \right\rangle \\
&= \left\langle \sum_{ij} \psi_i^{(A)}\psi_j^{(A)\dagger}\psi_j^{(B)}\psi_i^{(B)\dagger} + \sum_i \psi_i^{(A)}\psi_i^{(A)\dagger} \sum_k \psi_k^{(B)}\psi_k^{(B)\dagger} \right\rangle \\
&= \left\langle \sum_{ij} \psi_i^{(A)}\psi_j^{(A)\dagger}\psi_j^{(B)}\psi_i^{(B)\dagger} \right\rangle + 1
\end{aligned}$$

since wave functions are normalized.

The original Born rule can be written as

$$\begin{aligned}
P(\psi^{(A)} \to \psi^{(B)}) &= |\langle \psi^{(A)} | \psi^{(B)} \rangle|^2 \\
&= \sum_i \psi_i^{(A)}\psi_i^{(B)*} \sum_j \psi_j^{(A)*}\psi_j^{(B)} \\
&= \sum_{ij} \psi_i^{(A)}\psi_j^{(A)*}\psi_j^{(B)}\psi_i^{(B)*}
\end{aligned}$$

The previous expression $\left\langle \sum_{ij} \psi_i^{(A)}\psi_j^{(A)\dagger}\psi_j^{(B)}\psi_i^{(B)\dagger} \right\rangle$ represents the same calculation, with the extra step of taking the real part in the end. But as the probability is real anyway, we conclude that

$$P(A \to B) = \left\langle \Psi^{(A)}\Psi^{(A)\dagger}\Psi^{(B)}\Psi^{(B)\dagger} \right\rangle - 1$$

The initial geometric algebra expression for the wave vector was found by requiring this result (up to an arbitrary scalar).

B Derivation of state vector

We start with the expression

$$\Psi = \sum_i \frac{1}{\sqrt{2}}(e_i + Jf_i)(\Re\psi_i + J\,\Im\psi_i)$$

With $\psi_i = \Re\psi_i + J\,\Im\psi_i$ we can write

$$\begin{aligned}
\Psi\Psi^\dagger &= \frac{1}{2}\sum_{ij}(e_i + Jf_i)\psi_i\psi_j^\dagger(e_j - Jf_j) \\
&= \sum_i (1 - e_i f_i J)\psi_i\psi_i^\dagger + \frac{1}{2}\sum_{i<j}\Big((e_i + Jf_i)(e_j - Jf_j)\psi_i\psi_j^\dagger + (e_j + Jf_j)(e_i - Jf_i)\psi_j\psi_i^\dagger\Big) \\
&= \sum_i (1 - e_i f_i J)\psi_i\psi_i^\dagger \\
&\quad + \frac{1}{2}\sum_{i<j}\left(e_i e_j + f_i f_j - (e_i f_j + e_j f_i)J\right)\psi_i\psi_j^\dagger \\
&\quad + \frac{1}{2}\sum_{i<j}\left(e_j e_i + f_j f_i - (e_j f_i + e_i f_j)J\right)\psi_j\psi_i^\dagger
\end{aligned}$$

With the real and imaginary parts

$$R_{ij} = \frac{1}{2}(\psi_i\psi_j^\dagger + \psi_j\psi_i^\dagger) \qquad \psi_i\psi_j^\dagger = R_{ij} + J\,I_{ij}$$
$$JI_{ij} = \frac{1}{2}(\psi_i\psi_j^\dagger - \psi_j\psi_i^\dagger) \qquad \psi_j\psi_i^\dagger = R_{ij} - J\,I_{ij}$$

where R_{ij}, I_{ij} are scalars this becomes

$$\begin{aligned}\Psi\Psi^\dagger &= \sum_i (1 - e_i f_i J)\psi_i\psi_i^\dagger \\ &+ \frac{1}{2}\sum_{i<j}(e_ie_j + f_if_j - (e_if_j + e_jf_i)J)\,(R_{ij} + J\,I_{ij}) \\ &+ \frac{1}{2}\sum_{i<j}(-e_ie_j - f_if_j - (e_if_j + e_jf_i)J)\,(R_{ij} - J\,I_{ij}) \\ &= \sum_i \psi_i\psi_i^\dagger - \sum_i e_if_iJ\psi_i\psi_i^\dagger - \sum_{i<j}(e_if_j + e_jf_i)JR_{ij} + \sum_{i<j}(e_ie_j + f_if_j)J\,I_{ij}\end{aligned}$$

For normalized wave vectors

$$\sum_i \psi_i\psi_i^\dagger = 1$$

Therefore

$$\Omega = J(\Psi\Psi^\dagger - 1) = \sum_i e_if_i\psi_i\psi_i^\dagger + \sum_{i<j}(e_if_j + e_jf_i)R_{ij} - \sum_{i<j}(e_ie_j + f_if_j)\,I_{ij}$$

is a bivector which contains all the information that we need about a system to calculate it's probabilities.

C Derivation of spin state vector

The wavefunction for a single spin-up in a direction given by Euler angles θ, ϕ is usually written as

$$\psi = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} = \begin{pmatrix}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi}\end{pmatrix}$$

up to an arbitrary phase.

The state vector written in geometric algebra is

$$\begin{aligned}\Omega &= J(\Psi\Psi^\dagger - 1) \\ &= \sum_i e_if_i\,|\psi_i|^2 + \sum_{i<j}(e_if_j + e_jf_i)\,\Re(\psi_i\psi_j^*) - \sum_{i<j}(e_ie_j + f_if_j)\,\Im(\psi_i\psi_j^*) \\ &= e_1f_1\cos^2\frac{\theta}{2} + e_2f_2\sin^2\frac{\theta}{2} + (e_1f_2 + e_2f_1)\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\phi + (e_1e_2 + f_1f_2)\cos\frac{\theta}{2}\sin\frac{\theta}{2}\sin\phi \\ &= e_1f_1\frac{1-\cos\theta}{2} + e_2f_2\frac{1+\cos\theta}{2} + (e_1f_2 + e_2f_1)\,\frac{1}{2}\sin\theta\cos\phi + (e_1e_2 + f_1f_2)\,\frac{1}{2}\sin\theta\sin\phi \\ &= \frac{1}{2}(e_1f_1 + e_2f_2) + \frac{1}{2}(e_2f_2 - e_1f_1)\cos\theta + \frac{1}{2}\,(e_1f_2 + e_2f_1)\sin\theta\cos\phi + \frac{1}{2}\,(e_1e_2 + f_1f_2)\sin\theta\sin\phi\end{aligned}$$

D Multiplication of state vector basis

$$X = \frac{1}{2}\,(e_1f_2 + e_2f_1) \qquad Y = \frac{1}{2}\,(e_1e_2 + f_1f_2) \qquad Z = \frac{1}{2}(e_2f_2 - e_1f_1) \qquad T = \frac{1}{2}(e_1f_1 + e_2f_2)$$
$$P = e_1f_1e_2f_2$$

$$\begin{array}{llll} XX = -\frac{1}{2}(1+P) & YY = -\frac{1}{2}(1+P) & ZZ = -\frac{1}{2}(1+P) & TT = -\frac{1}{2}(1-P) \\ XP = X & YP = Y & ZP = Z & TP = -T \\ XY = Z & YZ = X & ZX = Y & \\ XT = 0 & YT = 0 & ZT = 0 & \end{array}$$

The bivectors X, Y, Z, T have no inverse.