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Exact Solution of the Problem of Brachistochrone with Allowance for the Coulomb Friction Forces

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Abstract—A system of differential equations describing the problem of brachistochrone with allowance for Coulomb friction forces is derived using the moving-basis method. An exact analytical solution of this system is obtained in the parametric form; a particular case of this solution in the limit $\mu \to 0$ is transformed into the equation of classical brachistochrone.

Keywords: brachistochrone, Coulomb friction force, moving basis, reaction force

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1. INTRODUCTION

Although the problem of brachistochrone, which was stated by J. Bernoulli in 1696, has been developed for more than three centuries and many studies in this field have been published, an exact analytical solution to this problem with allowance for the Coulomb friction force has not been found to the best of our knowledge (see, for example, [1–8]).

The purpose of this Letter is to fill this small gap in the theory of brachistochrone in a nonideal case in which the Coulomb friction force is also taken into account.

We assume that the system under study (a chute + a sphere rolling over it without slip under gravity) is in vacuum and the sphere is decelerated only by the force of interaction with the chute according to the Coulomb law. A schematic of the forces and geometric statement of the problem are given in Fig. 1.

It should be stated that brachistochrone properties are generally investigated using four main approaches: variational calculus principles [1–5], the theory of optimal control [9], the moving-basis method [10, 11], and an approach based on the solution of integral equations [12].

We present a solution of this problem using the moving-basis method, which is widely used and has been well approbated when solving a number of problems related to studying brachistochrone properties ([10, 11]).

PROBLEM SOLUTION

As can be seen in Fig. 1, the equation of motion can always be written as Newton's second law:

$$m\mathbf{a} = \mathbf{F}_{\text{fr}} + \mathbf{N} + m\mathbf{g}, \quad (1)$$

where m is the sphere mass, $\mathbf{F}_{\text{fr}}$ is the Coulomb friction force, $\mathbf{N}$ is the chute reaction force, and $\mathbf{g}$ is the acceleration of gravity.

At curvilinear motion, the acceleration is

$$\mathbf{a} = \dot{\mathrm{v}}\,\boldsymbol{\tau} + \frac{\mathrm{v}^2}{R}\mathbf{n}, \quad (2)$$

where $\boldsymbol{\tau} - \mathbf{n}$ is the moving basis shown in Fig. 1, R is the radius of curvature of the motion path, $\boldsymbol{\tau}$ is the unit vector of the tangent directed along the velocity, and $\mathbf{n}$ is the normal vector. From here on, a dot above a variable means time differentiation, i.e., $\dot{\mathrm{v}} = \frac{d\,\mathrm{v}}{dt}$, and it is assumed that $\mathrm{v} = \mathrm{v}(t)$.

Since

$$\mathbf{F}_{\text{fr}} = -F_{\text{fr}}\boldsymbol{\tau}, \quad \mathbf{N} = \mathbf{n}N, \quad \mathbf{g} = g(\boldsymbol{\tau}\sin\alpha + \mathbf{n}\cos\alpha), \quad (3)$$

where obtuse angle $\alpha(t)$ belongs to the segment $\alpha \in \left[\frac{\pi}{2}, \pi\right]$, with allowance for (2) and (3), Eq. (1) can be written as

$$m\left(\dot{\mathrm{v}}\,\boldsymbol{\tau} + \frac{\mathrm{v}^2}{R}\mathbf{n}\right) = -F_{\text{fr}}\boldsymbol{\tau} + N\mathbf{n} + mg(\boldsymbol{\tau}\sin\alpha + \mathbf{n}\cos\alpha). \quad (4)$$

Having projected this equation to the basis $\boldsymbol{\tau} - \mathbf{n}$, we arrive at the following system of equations:

$$\begin{cases} \dot{\mathrm{v}} = g\sin\alpha - \dfrac{F_{\text{fr}}}{m}, \\ \dfrac{m\,\mathrm{v}^2}{R} = N + mg\cos\alpha. \end{cases} \quad (5)$$

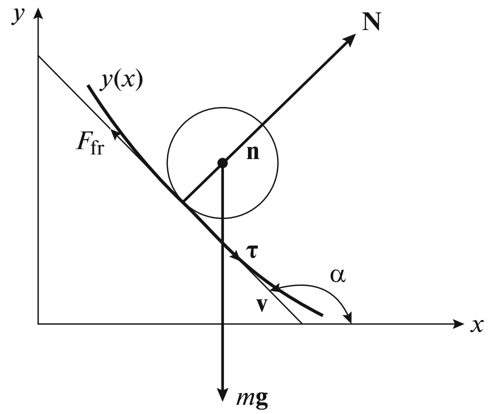

Fig. 1. Schematic image of the statement of the problem.

It follows from the lower equation that the reaction force is

$$N = m\left(\frac{v^2}{R} - g\cos\alpha\right). \quad (6)$$

When considering brachistochrone, the following condition should be satisfied (as was demonstrated in [12]):

$$\frac{v^2}{R} = -g\cos\alpha. \quad (7)$$

(Note that, if $\frac{v^2}{R} = g\cos\alpha$, the trajectory is a parabola.) Therefore, according to (6), the reaction force is

$$N = -2mg\cos\alpha \geq 0. \quad (8)$$

Therefore, the friction force is

$$F_{fr} = \mu N = -2\mu mg\cos\alpha. \quad (9)$$

This means that system of equations (5), with allowance for (7) and (9), can be written in the form

$$\begin{cases} \dot{v} = g(\sin\alpha + 2\mu\cos\alpha), \\ \frac{v^2}{R} = -g\cos\alpha. \end{cases} \quad (10)$$

Since

$$v = R\dot{\alpha}, \quad (11)$$

we arrive at the following system (instead of (10)), which is much simpler:

$$\begin{cases} \dot{v} = g(\sin\alpha + 2\mu\cos\alpha), \\ v\dot{\alpha} = -g\cos\alpha. \end{cases} \quad (12)$$

Having divided the upper equation by the lower, we obtain

$$\frac{dv}{v\,d\alpha} = -\tan\alpha - 2\mu.$$

After elementary integration, we find that

$$v = C_1|\cos\alpha|e^{-2\mu\alpha},$$

where C_1 is the integration constant. Since angle α is obtuse, we have

$$v = -C_1\cos\alpha e^{-2\mu\alpha}. \quad (13)$$

Afterwards, using the equations

$$\dot{x} = -v\cos\alpha \geq 0,$$
$$\dot{y} = -v\sin\alpha,$$

we directly obtain

$$\begin{cases} x(\alpha) = -\int_{\frac{\pi}{2}}^{\alpha} \frac{v(\varphi)\cos\varphi}{\dot{\varphi}}d\varphi, \\ y(\alpha) = H - \int_{\frac{\pi}{2}}^{\alpha} \frac{v(\varphi)\sin\varphi}{\dot{\varphi}}d\varphi. \end{cases} \quad (14)$$

Since, according to Eqs. (12),

$$v\dot{\alpha} = -g\cos\alpha,$$

we find with allowance for solution (13) that

$$\dot{\alpha} = \frac{g}{C_1}e^{2\mu\alpha}. \quad (15)$$

A solution to this equation can be written as

$$\int_{\frac{\pi}{2}}^{\alpha} e^{-2\mu\varphi}d\varphi = \frac{g}{C_1}\int_0^t d\xi,$$

where it is taken into account that $\alpha(t)|_{t=0} = \frac{\pi}{2}$. Hence,

$$\frac{1}{2\mu}\left(e^{-\pi\mu} - e^{-2\mu\alpha}\right) = \frac{gt}{C_1}. \quad (16)$$

Substituting now (15) and (13) into solution (14), we obtain

$$\begin{cases} x(\alpha) = \frac{C_1^2}{g}\int_{\frac{\pi}{2}}^{\alpha} e^{-4\mu\varphi}\cos^2\varphi d\varphi, \\ y(\alpha) = H + \frac{C_1^2}{2g}\int_{\frac{\pi}{2}}^{\alpha} e^{-4\mu\varphi}\sin 2\varphi d\varphi. \end{cases} \quad (17)$$

It follows from the law of conversation of energy

$$\frac{m(\dot{x}^2 + \dot{y}^2)}{2} + mgy = mgH = \text{const}$$

that

$$C_1 = \sqrt{2gH}. \quad (18)$$

After simple integration, we arrive at the final analytical solution of the problem of brachistochrone with allowance for the Coulomb friction force:

$$\begin{cases} x(\alpha) = \dfrac{H}{2}\left\{\dfrac{1}{2\mu}\left(e^{-2\pi\mu} - e^{-4\mu\alpha}\right) + \dfrac{1}{1+4\mu^2}\left[e^{-4\mu\alpha}(\sin 2\alpha - \mu\cos 2\alpha) - \mu e^{-2\pi\mu}\right]\right\}, \\ y(\alpha) = H\left\{1 - \dfrac{1}{2\left(1+4\mu^2\right)}\left[e^{-2\pi\mu} + e^{-4\mu\alpha}(\cos 2\alpha + 2\mu\sin 2\alpha)\right]\right\}. \end{cases} \quad (19)$$

It follows from solution (19) that, in the ideal-case limit (where the friction coefficient is $\mu \to 0$), we obtain the parametric equation of classical brachistochrone

$$\begin{cases} x(\alpha) = H\left(\alpha + \dfrac{1}{2}\sin 2\alpha - \dfrac{\pi}{2}\right), \\ y(\alpha) = \dfrac{H}{2}(1 - \cos 2\alpha). \end{cases} \quad (20)$$

The fact that solution (20) describes specifically brachistochrone can easily be verified by calculation of the time of ball rolling from the upper chute point (where $\alpha = \frac{\pi}{2}$) to the lower one (where $\alpha = \pi$). For the classical brachistochrone, this time is

$$\Delta t = \pi\sqrt{\frac{H}{2g}}. \quad (21)$$

Indeed, according to solution (20), we have

$$\Delta t = \int_{\frac{\pi}{2}}^{\pi} \frac{\sqrt{\left(\frac{dx}{d\alpha}\right)^2 + \left(\frac{dy}{d\alpha}\right)^2}}{v} d\alpha = -2H\int_{\frac{\pi}{2}}^{\pi} \frac{\cos\alpha d\alpha}{v}.$$

Taking into account expression (13), according to which $v = -C_1\cos\alpha$, we find directly the desired rolling time:

$$\Delta t = \pi\sqrt{\frac{H}{2g}},$$

which coincides with (21), as was to be proved.

CONCLUSIONS

(i) Equations of slip-free motion of a ball over a chute with allowance for the Coulomb friction force were obtained using the moving-basis method.

(ii) An exact solution of the problem in parametric form was found for the case of $\mu \neq 0$.

CONFLICT OF INTEREST

The author declares that he has no conflicts of interest.

REFERENCES

1. N. M. Günther, *The Course of the Calculus of Variations* (Gosthekhizdat, Moscow, 1941) [in Russian].
2. M. A. Lavrent'ev and L. A. Lyusternik, *The Course of the Calculus of Variations,* 2nd ed. (Gosthekhizdat, Moscow, 1950) [in Russian].
3. I. M. Gelfad and S. V. Fomin, *Calculus of Variations* (Fizmatgiz, Moscow, 1961; Dover, New York, 1963).
4. V. I. Smirnov, *Course of Higher Mathematics* (Nauka, Moscow, 1967; Mir, Moscow, 1964), Vol. 4.
5. L. E. Elsgoltz, *Differential Equations and the Calculus of Variations* (Nauka, Moscow, 1969; Mir, Moscow, 1977).
6. N. Ashby, W. E. Brittin, W. F. Love, and W. Wyss, Am. J. Phys. **43**, 902 (1975).
7. S. C. Lipp, SIAM J. Control Optim. **35**, 562 (1997).
8. J. C. Nayen, Int. J. Non-Lin. Mech. **40**, 1057 (2005).
9. L. S. Pontryagin, V. G. Boltyansky, R. V. Gamkrelidze, and E. F. Mishchenko, *The Mathematical Theory of Optimal Processes* (Fizmatgiz, Moscow, 1961; Wiley, New York, 1962).
10. S. O. Gladkov, Uch. Zap. Fiz. Fak. MGU, No. 4, 164002-1 (2016).
11. S. O. Gladkov and S. B. Bogdanova, J. Math. Sci. **245**, 528 (2020).
12. A. A. Barsuk and F. Paladi, Int. J. Non-Lin. Mech. **148**, 43 (2023).

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