4   The Equivalence Principle

We are now in a position to *prove* the equivalence principle of General Relativity. Indeed as we will see, from our QFT viewpoint, the equivalence principle is not an independent principle. Rather, it is a necessary consequence of Lorentz invariance for massless spin-2 particles—a *theorem* then, but unfortunately the name 'equivalence theorem' is taken already! The equivalence principle was proved by Weinberg. We will follow his original paper [2], as well as his QFT book [1], in particular sect. 13.1.

4.1   Photon couplings: charge conservation

As a warmup we consider the consequences of Lorentz invariance for the interactions of a photon with other particles. We will show that a low-energy photon can only couple to charges that are conserved by all scattering processes.

Consider the emission of a very soft photon of four-momentum q^μ by another particle, massive or otherwise, as depicted in fig. 1. By 'soft' we mean that the photon's energy is very low. Implicitly we are taking the limit $q^\mu \to 0$, or more precisely, we are expanding in powers of q^μ, keeping the leading terms only. For definiteness let's restrict to a $+1$ helicity photon, although everything we say will apply equally to both photon helicities. The initial and final states of the emitting particle are labeled by their four-momenta, p^μ and $p^\mu - q^\mu$, and by their spin state, σ and σ'.

Of course the process of fig. 1 cannot happen for finite q^μ if all particles are on-shell, because of four-momentum conservation. But this diagram can be part of a larger diagram, where, say, only the initial particle line and the emitted photon are on-shell. Indeed this vertex diagram will be one of the fundamental units of our proof, which will involve more complicated diagrams in
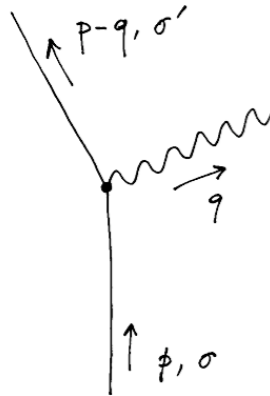

Figure 1: *The emission of a soft photon of momentum q^μ by a incoming particle with momentum p^μ and spin state σ.*

which our vertex appears as a sub-diagram.

The amplitude associated with our diagram takes the form

$$\mathcal{M} = M^\mu \cdot e_\mu^*(q) \, , \tag{96}$$

where $e_\mu(q)$ is the polarization vector of the emitted photon, and M^μ depends on the rest of the diagram, that is, on the final and initial states of the emitting particle. In the very soft limit, $q^\mu \to 0$, the initial and final momenta coincide, and we have

$$M^\mu = M^\mu(p, \sigma, \sigma') \tag{97}$$

We can now consider two separate cases:

- If the particle under study is spinless, i.e. if it is a *scalar* particle, there is no spin variable to take into account. M^μ then can only depend on p^μ. Since it must be a four vector, it takes the form

$$M^\mu(p) = p^\mu f(p^2) \, . \tag{98}$$

  If the initial particle is *on-shell*, i.e. if it is a physical particle coming in from infinity, rather than a virtual particle in an internal line in a diagram, then $p^2 = -m^2$. In this case $f(p^2) = f(-m^2)$ is a constant that depends on the particle species, but not on the kinematical variables of our process. We call this constant *the charge* of the particle. More precisely, with the conventions of ref. [1]

$$\mathcal{M} = 2i \, (2\pi)^4 \, e \, p^\mu \cdot e_\mu^*(q) \, , \qquad \text{for} \quad q^\mu \to 0 \, . \tag{99}$$

Example: scalar QED. To convince ourselves that this matches the usual definition of charge, let's consider scalar QED. The Lagrangian is

$$\mathcal{L} = \left(D_\mu \phi\right)^\dagger D^\mu \phi - m^2 \phi^\dagger \phi - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \, , \tag{100}$$

where the covariant derivative is

$$D_\mu \equiv \partial_\mu - ie \, A_\mu \tag{101}$$

The amplitude we are interested in receives contributions from the Lagrangian terms that involve one ϕ (to destroy the incoming scalar), one $\phi^\dagger$ (to create the outgoing scalar), and one A_μ (to create the outgoing photon). By expanding the covariant derivatives we get the relevant Lagrangian terms

$$\mathcal{L} \supset ie \, A^\mu \big(\phi^\dagger \partial_\mu \phi - (\partial_\mu \phi^\dagger)\phi\big) \tag{102}$$

The amplitude for fig. 1 is then

$$\mathcal{M} = i(2\pi)^4 \, e \left[p^\mu e_\mu^*(q) + (p^\mu - q^\mu)e_\mu^*(q)\right] \tag{103}$$

which for $q^\mu \to 0$ reduces precisely to eq. (99).

- If the emitting particle has spin $J \neq 0$, the amplitude for our elementary process in fig. 1 may be more complicated, because now M^μ can depend on the momenta as well as on the spin variables. Namely, unlike in eq. (98), the direction of M^μ now needs not be along p^μ, because there are other four-vectors that we can construct employing the spin variables. For instance, if the particle under study is a massive spin one particle, we have the polarization four-vectors $e_\sigma^\mu(p)$ and $e_{\sigma'}^\mu(p)$. Also, even if M^μ does point along p^μ, the overall coefficient— the analogue of $f(p^2)$ in eq. (98)—may in principle depend on all the Lorentz invariants we can construct with the momenta and spin variables. Let us *postulate* for the moment that even for particles with spin, the amplitude for emitting a very soft photon does *not* depend on the emitting particle's spin, and that the final spin σ' has to be the same as the initial one:

$$\mathcal{M} = 2i \, (2\pi)^4 \, e \, p^\mu \cdot e_\mu^*(q) \cdot \delta_{\sigma\sigma'} \,, \qquad \text{for} \quad q^\mu \to 0 \,. \tag{104}$$

  We will question this assumption later, and see why it is in fact necessary for Lorentz invariance to hold.

Consider now a generic scattering process $\alpha \to \beta$, where several particles come in from infinity, collide, and then the same or other particles go out to infinity, as schematically depicted in fig. 2a. The 'blob' represents the full scattering amplitude—the sum of all Feynman diagrams with initial state α and final state β. Let us call $\mathcal{M}_{\alpha\beta}$ this scattering amplitude. Now, consider exactly the same process where a very soft photon of four-momentum q^μ is also emitted, as in fig. 2b. (Of course for finite q^μ the rest of the external momenta cannot be exactly the same as in $\alpha \to \beta$, because we could not conserve four-momentum unless we slightly modified some of the initial or final momenta; but in the $q^\mu \to 0$ limit we do approach the original momenta of the $\alpha \to \beta$ process.) If we expand this amplitude in Feynman diagrams, Feynman rules instruct us to take all the diagrams that make up the original $\mathcal{M}_{\alpha\beta}$, and attach the photon line to all possible particle lines, external as well as internal.

Let's focus on the diagrams where the photon line is attached to an *external* line, as in fig. 3a. The new vertex involving the photon is just our elementary vertex of fig. 1, and gives a factor of eq. (104). Because this is proportional to $\delta_{\sigma\sigma'}$, after emitting the photon particle 1 still has its original spin σ_1. Also, the photon is very soft, so particle 1 retains its orginal momentum p_1 too. As a result, the rest of the process of fig. 3a is *identical* to the original $\alpha \to \beta$ process, with precisely the same external momenta and spins. We therefore get a factor of $\mathcal{M}_{\alpha\beta}$. The last ingredient is just the propagator that connects the photon vertex to the rest of the diagram, with momentum $p_1 - q$. Putting everything together we have

$$\text{fig. 3a} = \mathcal{M}_{\alpha\beta} \times \frac{(-i)}{(2\pi)^4} \frac{1}{(p_1 - q)^2 + m_1^2} \times 2i \, (2\pi)^4 \, e_1 \, p_1^\mu \cdot e_\mu^*(q) \,, \tag{105}$$

where m_1 and e_1 are particle 1's mass and charge, respectively, and we are neglecting the $+i\epsilon$ in the propagator, because it is irrelevant for our purposes. It is crucial that we keep $q^\mu \neq 0$ in the propagator. This is because the propagator is singular in the $q^\mu \to 0$ limit. Indeed

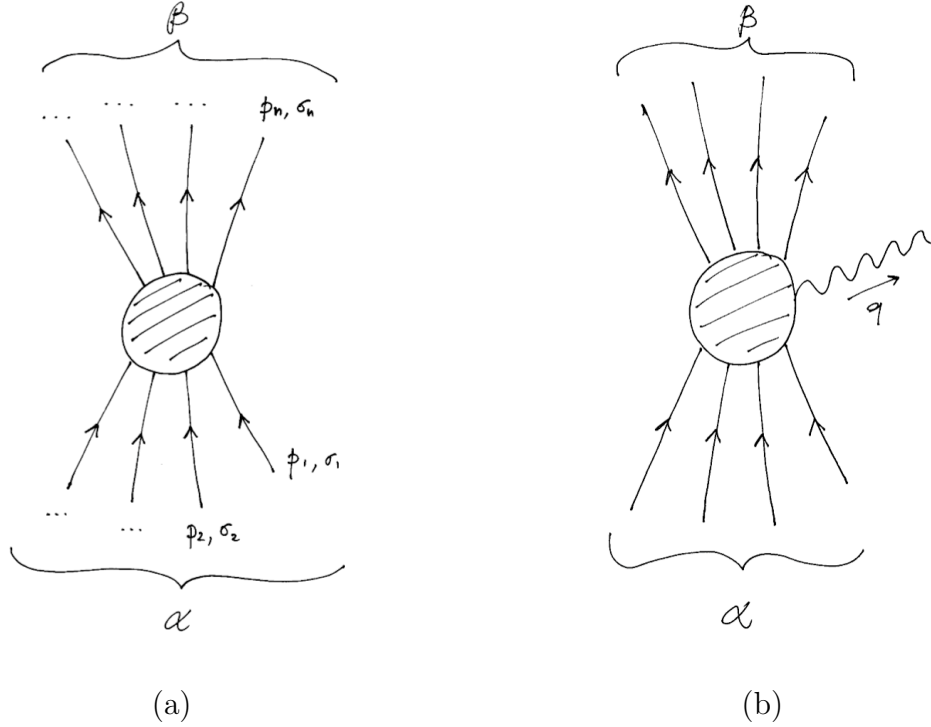$$(p_1 - q)^2 + m_1^2 = p_1^2 - 2p_1 \cdot q + q^2 + m_1^2 = -2p_1 \cdot q \,, \tag{106}$$

Figure 2: (a): *A generic scattering process with initial state α and final state β.* (b): *The same process where a very soft photon is also emitted.*

where we used that both the initial particle 1 and the emitted photon are on-shell. Therefore we have

$$\text{fig. 3}a = \mathcal{M}_{\alpha\beta} \times \frac{-e_1 \, p_1^\mu}{p_1 \cdot q} \, e_\mu^*(q) \; . \tag{107}$$

We get analogous contributions from the diagrams in which the photon is attached to a different external particle line, with the qualification that for outgoing particle lines we get an overall plus rather than a minus. This is because if we attach the photon to an outgoing line with momentum p_n^μ, the momentum flowing in the propagator *before* the photon is emitted is $p_n^\mu + q^\mu$, as clear from fig. 3b. Then the full amplitude for our process of fig. 2b is conveniently rewritten as

$$\mathcal{M} = M^\mu \, e_\mu^*(q) \tag{108}$$

with

$$M^\mu = \mathcal{M}_{\alpha\beta} \sum_n \frac{\eta_n \, e_n \, p_n^\mu}{p_n \cdot q} + \text{internal} \; , \tag{109}$$

where the sum runs over all external particles, η_n is a sign that is $+1$ for outgoing particles and -1 for incoming ones, and 'internal' stands for the sum of all the diagrams where the photon is attached to an internal particle line. However the latter contributions are not singular in the $q^\mu \to 0$ limit. Indeed the propagators we isolated above are singular precisely because the external

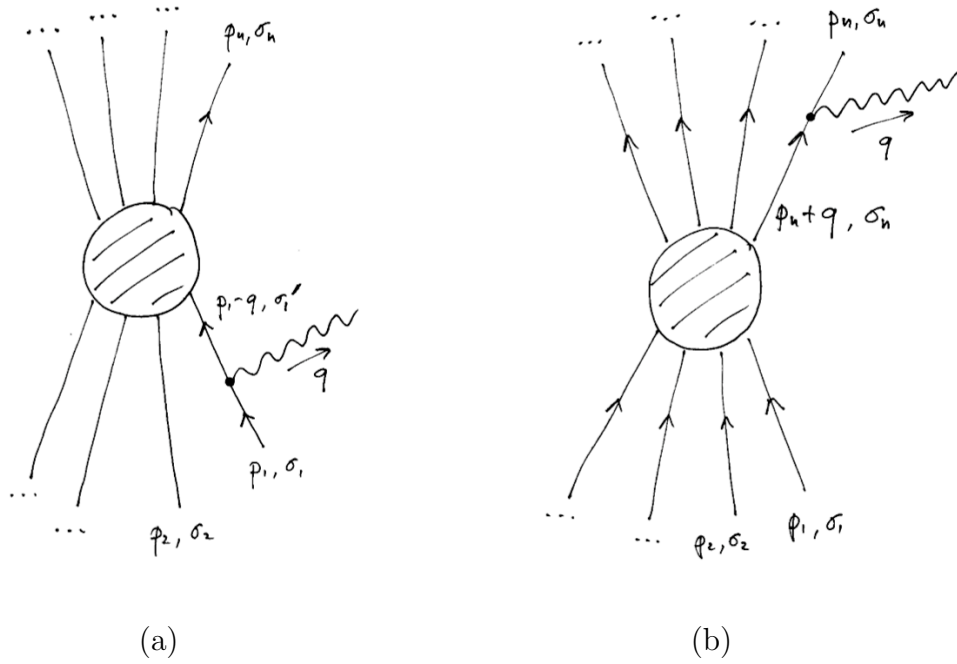(a)   (b)

Figure 3: (a): *A contribution to the amplitude of fig. 2b where the photon is emitted by an incoming external particle.* (b): *The analogous contribution for an outgoing external particle.*

particles are on-shell, so that when we send $q \to 0$ the momentum flowing in the propagator goes on-shell, thus making the propagator diverge. Instead virtual particles are not on-shell, and as a consequence attaching a photon to their lines introduces a new propagator that does not go on-shell when $q^\mu \to 0$. Since divergences can only come from propagators, and not from vertices—which are just products of coupling constants, polarization spinors/vectors/tensors, and derivatives— we conclude that the 'internal' piece in M^μ above is not singular for $q^\mu \to 0$, and is therefore subleading in a small q expansion. We will therefore neglect it.

We now make use of Lorentz invariance. More precisely, we make use of the the redundancy in describing the emitted photon with its polarization four-vector: $e_\mu(q)$ and $e_\mu(q) + c\,q_\mu$ must describe the same photon state, for any complex c. As a consequence, when computing a physical amplitude like the one under study we must have

$$M^\mu q_\mu = 0 \ . \tag{110}$$

We can call this 'gauge-invariance of the S-matrix', although as we stressed the only real symmetry we are trying to enforce is Lorentz invariance, and gauge invariance is a formal redundancy that allows us to construct Lorentz invariant theories. From eq. (135) we have

$$\mathcal{M}_{\alpha\beta} \sum_n \eta_n\, e_n = 0 \ , \tag{111}$$

that is, either the original amplitude $\mathcal{M}_{\alpha\beta}$ vanishes, in which case the process $\alpha \to \beta$ cannot happen, or such a process *must conserve charge*:

$$\sum_{\text{incoming}} e_n = \sum_{\text{outgoing}} e_n \,. \tag{112}$$

In other words, a very low energy photon can only couple to a charge that is conserved by all scattering processes that have non-vanishing probability of happening.

We can now go back to the crucial assumption of our proof—that in a vertex like fig. 1, even for particles with spin, the amplitude for emitting a very soft photon is independent of the initial spin σ, and is non-vanishing only if the final spin σ' coincides with σ, as in eq. (104). Let us see what would happen if this were not the case. As we argued, in the small q^μ limit the most generic form of a vertex like fig. 1 is

$$\mathcal{M} = M^\mu(p, \sigma, \sigma') \cdot e_\mu^*(q) \,, \tag{113}$$

where M^μ is a four-vector function of its arguments. We can go through our computation above for the amplitude $\alpha \to \beta +$ photon, and see the changes that would occur if we where to use this elementary vertex amplitude. When we attach the photon line to an external particle line like in fig. 3a, we have two differences: the vertex amplitude of course is different, but also the particle line that connects the vertex to the blob now has an arbitrary spin state σ_1', over which we have to sum. As a consequence, in fig. 3a we do not get an overall factor of the original amplitude $\mathcal{M}_{\alpha\beta}$, but rather $\mathcal{M}_{\alpha\beta}(\ldots, \sigma_1', \ldots)$—the amplitude for the $\alpha \to \beta$ process with particle 1's spin σ_1 replaced by σ_1'. Then eq. (107) becomes

$$\text{fig. 3a} = \sum_{\sigma_1'} \mathcal{M}_{\alpha\beta}(\ldots, \sigma_1', \ldots) \times \frac{-M_1^\mu(p_1, \sigma_1, \sigma_1')}{p_1 \cdot q} \, e_\mu^*(q) \,, \tag{114}$$

where we are neglecting the overall factors of i, (2π), etc. we would get from the propagator. Of course we get an analogous contribution from each external particle line, whereas diagrams where the photon line is attached to an internal particle line of the $\alpha \to \beta$ process still give negligible contributions, because with respect to our proof above we are just changing the structure of vertices, while as we argued divergences in the $q^\mu \to 0$ limit can only come from propagators. Therefore, the overall amplitude for emitting a very soft photon in the $\alpha \to \beta$ process now is

$$\mathcal{M} = \sum_n \sum_{\sigma_n'} \mathcal{M}_{\alpha\beta}(\ldots, \sigma_n', \ldots) \frac{\eta_n \, M_n^\mu(p_n, \sigma_n, \sigma_n')}{p_n \cdot q} \, e_\mu^*(q) \tag{115}$$

Because of Lorentz-invariance, this amplitude must vanish if we formally replace the emitted photon's polarization four-vector with the photon's momentum q^μ:

$$\sum_n \sum_{\sigma_n'} \mathcal{M}_{\alpha\beta}(\ldots, \sigma_n', \ldots) \frac{\eta_n \, M_n^\mu(p_n, \sigma_n, \sigma_n') q_\mu}{p_n \cdot q} = 0 \,. \tag{116}$$

Now, notice that for given p_n's and σ_n's, the l.h.s. is a smooth function of the photon's four-vector q^μ—apart from the divergences at the poles of course. If it has to vanish for all light-like four-momenta q^μ, it must also vanish for an arbitrary four-vector (not necessarily light-like) q^μ. This is only possible if the direction of each individual $M_n^\mu(p_n, \sigma_n, \sigma_n')$ is along p_n^μ. Indeed, for any given external particle n, we can choose the four-vector q^μ to be orthogonal to p_n^μ, but not orthogonal to the other external four-momenta. The n-th term in eq. (116) is therefore divergent *at finite* q^μ, and cannot be cancelled by the other terms which are instead finite. The cancellation must therefore come from its numerator, that is

$$M_n^\mu(p_n, \sigma_n, \sigma_n')q_\mu = 0 \qquad \forall q^\mu \text{ such that } \quad q_\mu p_n^\mu = 0 , \tag{117}$$

that is,

$$M_n^\mu(p_n, \sigma_n, \sigma_n') = p_n^\mu \times e_n(p_n, \sigma_n, \sigma_n') . \tag{118}$$

This looks like eq. (104), but with a momentum- and spin-dependent charge. If we plug it back into eq. (116) we are left with

$$\sum_n \sum_{\sigma_n'} \mathcal{M}_{\alpha\beta}(\dots, \sigma_n', \dots) \times \eta_n \, e_n(p_n, \sigma_n, \sigma_n') = 0 . \tag{119}$$

This should hold for *all* processes $\alpha \to \beta$. Now, the problem with this equation, as opposed to eq. (111), is that rather than constraining the way the photon couples to other particles, it is a formidable constraint on the amplitude for the original process $\alpha \to \beta$ (with some of the spins modified). In particular, it is a constraint that *depends* on the way the photon couples to other particles, i.e. on the 'charge functions' $e_n(p_n, \sigma_n, \sigma_n')$. But, at least at lowest order in perturbation theory, the original process $\alpha \to \beta$ knows nothing about the photon and its couplings, and the only constraint it satisfies is four-momentum conservation. Thus, there is no way of satisfying eq. (119) *unless*

$$e_n(p_n, \sigma_n, \sigma_n') = e_n(p_n, \sigma_n) \times \delta_{\sigma_n \sigma_n'} , \tag{120}$$

so that the sum over σ_n' is trivial and an overall $\mathcal{M}_{\alpha\beta}$ gets factored out of the sum over n:

$$\mathcal{M}_{\alpha\beta} \sum_n \eta_n \, e_n(p_n, \sigma_n) = 0 . \tag{121}$$

We therefore get that for processes with non-vanishing amplitude, the momentum- and spin-dependent charges $e_n(p_n, \sigma_n)$ must be conserved. But if the process $\alpha \to \beta$ has a non-vanishing amplitude, the same process where we change, say, particle 1's spin to σ_1' and we leave everything else unchanged also has—barring accidents for very specific configurations of momenta and spins—non-vanishing amplitude: once again, the only *kinematical* constraint for scattering amplitudes is the total four-momentum δ-function [4]. Therefore, the momentum- and spin-dependent charges

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[4]Of course for massless particles of definite helicities we might have the situation where only one of the two helicities interact. This is the case for instance for left-handed massless neutrinos in the Standard Model of electroweak interactions. But as we saw, different helicities of a massless particle strictly speaking correspond to different particles, so changing helicity of a massless particle is not merely a change of our spin label σ_n.

must be conserved by the two processes:

$$-e_1(p_1, \sigma_1) + \sum_{n>1} \eta_n \, e_n(p_n, \sigma_n) = 0 = -e_1(p_1, \sigma_1') + \sum_{n>1} \eta_n \, e_n(p_n, \sigma_n) \,, \tag{122}$$

from which we get

$$e_1(p_1, \sigma_1) = e_1(p_1, \sigma_1') \,, \tag{123}$$

that is, the charge of a particle cannot depend on its spin state. But then, if e_n is a scalar function of p_n^μ only, for on-shell particles it is a constant, because $p_n^2 = -m_n^2$. We therefore reach the conclusion that even for particles with spin, the amplitude for emitting a very soft photon is given by eq. (104), where e is a constant that depends on the particle species only, and that any process with non-vanishing amplitude must conserve the total charge.

4.2 Graviton couplings: the equivalence principle

With minor modifications, the same argument can be applied to the graviton. Here however the constraint we get will be much stronger. Consider the emission of a very soft graviton by some particle, as depicted in fig. 1. The amplitude takes the form

$$\mathcal{M} = M^{\mu\nu} \, e_{\mu\nu}^*(q) \,, \tag{124}$$

where $e_{\mu\nu}(q)$ is the polarization tensor for the outgoing graviton, and the 'tensor amplitude' $M^{\mu\nu}$ is a function of the kinematical variables of the other particle: p, σ and $p - q, \sigma'$. In the very soft limit p and $p - q$ coincide, and we are left with

$$M^{\mu\nu} = M^{\mu\nu}(p, \sigma, \sigma') \,, \qquad q \to 0 \,. \tag{125}$$

Now, like for the photon, if the emitting particle is spinless, $M^{\mu\nu}$ can only depend on p; it must be a tensor; the only possibility is

$$M^{\mu\nu} = p^\mu p^\nu \, f(p^2) \,. \tag{126}$$

But if the emitting particle is on-shell, $p^2 = -m^2$ is a constant. Therefore $f(p^2)$ is a constant, which only depends on the particle species but not on its state. This defines the gravitational coupling constant for the particle under study. More precisely, with our conventions,

$$\mathcal{M} = 2i \, (2\pi)^4 f \, p^\mu p^\nu \cdot e_{\mu\nu}^*(q) \,, \qquad \text{for} \quad q^\mu \to 0 \,. \tag{127}$$

Example: scalar field in GR. Of course our goal is to *prove* that general relativity is essentially the only possible theory of gravitation, but let's consider it here as an example, to check whether eq. (127) makes sense. The action for gravity and a (minimally coupled) scalar ϕ in GR is

$$S = S_g + S_\phi = \tfrac{1}{16\pi G} \int d^4x \sqrt{-g} \, R + \int d^4x \sqrt{-g} \big[-\tfrac{1}{2} g^{\mu\nu} \, \partial_\mu \phi \, \partial_\nu \phi - \tfrac{1}{2} m^2 \phi^2 \big] \,. \tag{128}$$

We get non-trivial contributions to the amplitude of fig. 1 only from interactions that involve two ϕ fields (to create and destroy the two scalar particles in the diagram) and one $h_{\mu\nu}$ (to create the outgoing graviton). These only comes from S_ϕ, which we expand in powers of $h_{\mu\nu}$:

$$S_\phi[\phi, g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}] = S_\phi[\phi, \eta_{\mu\nu}] + \int d^4x \left.\frac{\delta S_\phi}{\delta g_{\mu\nu}(x)}\right|_{\eta_{\mu\nu}} h_{\mu\nu}(x) + \mathcal{O}(h^2) \ . \tag{129}$$

The linear piece—which is what interests us—is related to the ϕ stress-energy tensor. Indeed, given the definition of the latter

$$T^{\mu\nu}(x) = \frac{2}{\sqrt{-g}} \frac{\delta S_\phi}{\delta g_{\mu\nu}(x)} \ , \tag{130}$$

we have

$$S_\phi[\phi, g_{\mu\nu}] = S_\phi^{\text{flat}} + \int d^4x \, \tfrac{1}{2} h_{\mu\nu} T_{\text{flat}}^{\mu\nu} + \dots \ , \tag{131}$$

where 'flat' reminds us to compute the corresponding quantity for $g_{\mu\nu} = \eta_{\mu\nu}$. The stress-energy tensor of our scalar in flat space is

$$T_{\text{flat}}^{\mu\nu} = \partial^\mu \phi \, \partial^\nu \phi + \tfrac{1}{2} \eta^{\mu\nu} \left[-\partial_\alpha \phi \, \partial^\alpha \phi - m^2 \phi^2 \right] \ . \tag{132}$$

The amplitude for the process in fig. 1 is then

$$\mathcal{M} = i(2\pi)^4 \, \tfrac{1}{2} \left[(ip^\mu)\big(-i(p^\nu - q^\nu) \big) + (\mu \leftrightarrow \nu) \right] e_{\mu\nu}^*(q) + \# \, \eta^{\mu\nu} \, e_{\mu\nu}^*(q) \ . \tag{133}$$

The last piece vanishes, because the graviton polarization tensor is traceless (as well as purely spatial, and transverse). The rest, for $q^\mu \to 0$ reduces precisely to eq. (127), with gravitational coupling $f = \tfrac{1}{2}$.

Notice that we should not trust this particular value of f, because we have been cavalier about the normalization of the graviton field $h_{\mu\nu}$. In GR the metric is dimensionless, and so is $h_{\mu\nu}$. However we know that in QFT canonically normalized fields are typically dimensionful. In particular integer-spin fields have mass dimension one. This means that the correct $h_{\mu\nu}$ field to use in computing amplitudes differs from the GR one in the normalization. As a consequence eq. (133) is off by the same normalization correction factor. We can postpone for the moment the issue of the correct normalization of $h_{\mu\nu}$, since as long as we only consider processes where a single graviton is emitted or absorbed, the same correction factor will enter all amplitudes we compute as an overall constant. Ultimately, as we will see, the correct value of f is related to Newton's constant.

If the particle emitting the graviton has spin $J > 0$, in principle the tensor amplitude $M^{\mu\nu}$ can depend on the spin variables σ, σ' as well, and its tensor structure needs not be just $p^\mu p^\nu$. Like for the photon, we postulate that in the soft limit there is in fact no spin-dependence in $M^{\mu\nu}$, and that the amplitude is non-vanishing only if the initial spin is the same as the final one:

$$\mathcal{M} = 2i \, (2\pi)^4 \, f \, p^\mu p^\nu \cdot e_\mu^*(q) \cdot \delta_{\sigma\sigma'} \ , \qquad \text{for} \quad q^\mu \to 0 \ . \tag{134}$$

We will discuss later why this assumption is justified.

Now we follow precisely the same steps as for the photon. We consider a generic scattering process $\alpha \to \beta$ (fig. 2a), and the same process where a very soft graviton is also emitted (fig. 2b). The amplitude for the latter is the sum of all diagrams that make up the former where we also attach the new graviton line in all possible places. The only ingredient that differs from the photon case is our new elementary vertex for fig. 1, eq. (134). In particular all propagators are the same as for the photon emission case, and for $q \to 0$ we can neglect the diagrams where we attach the graviton to an internal particle line of the original $\alpha \to \beta$ process. We get

$$\text{fig. 2b} = \mathcal{M}_{\alpha\beta} \sum_n \frac{\eta_n\, f_n\, p_n^\mu p_n^\nu}{p_n \cdot q} \cdot e_{\mu\nu}^*(q) \,, \tag{135}$$

where the sum runs over all external legs, η_n is -1 and $+1$ for incoming and outgoing particles, respectively, and the f_n's are the individual gravitational coupling constants, which may in principle depend on the particle species.

Now, as we argued, for the amplitude to be Lorentz-invariant, it must vanish if we formally replace the graviton polarization tensor with q_μ. Imposing this we have

$$\mathcal{M}_{\alpha\beta} \sum_n \eta_n\, f_n\, p_n^\mu = 0 \,. \tag{136}$$

That is, either $\mathcal{M}_{\alpha\beta}$ vanishes, in which case the original process $\alpha \to \beta$ is forbidden, or we have discovered an additive four-vector

$$F^\mu \equiv \sum f_n\, p_n^\mu \tag{137}$$

that has to be conserved by all possible scattering processes:

$$F^\mu_{\text{incoming}} = F^\mu_{\text{outgoing}} \tag{138}$$

But the only such quantity conserved by all non-trivial scattering processes is the *total four-momentum*. By 'non-trivial' we mean processes where the individual particle momenta change. Then F^μ must be proportional to the total four-momentum P^μ, which is only possible if all different gravitational coupling constants f_n are in fact the same constant f, which is thus independent of the particle species,

$$f_n = f \qquad \forall \text{ particles} \,, \qquad F^\mu = f P^\mu \,. \tag{139}$$

To see that indeed this is the only possibility, consider an *elastic* $2 \to 2$ collision [5], where particles a and b—with gravitational couplings f_a and f_b—have initially momenta $\vec{p}_a$ and $\vec{p}_b$, collide, and emerge from the collision with momenta $\vec{p}_a{}'$ and $\vec{p}_b{}'$. In the CM frame we have conservation of the total momentum

$$\vec{p}_a + \vec{p}_b = 0 = \vec{p}_a{}' + \vec{p}_b{}' \tag{140}$$

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[5] By 'elastic' we denote those collisions that do not change the particle content of the initial state, but only their momenta and spins. That is, the particles entering the collision region are the same as those exiting it.
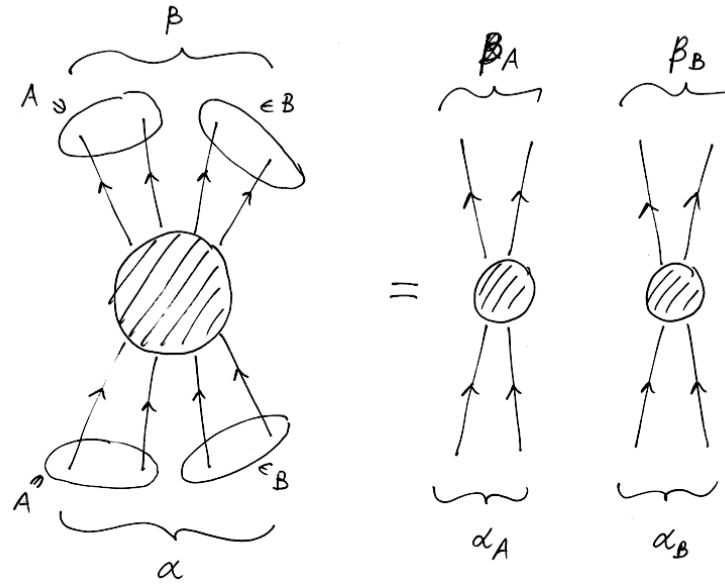
Figure 4: *If there are two sectors A, B of the world that do not interact with each other, each scattering amplitude factorizes into two sub-amplitudes.*

as well as of our new quantity F^μ:

$$f_a\, \vec{p}_a + f_b\, \vec{p}_b = f_a\, \vec{p}_a{}' + f_b\, \vec{p}_b{}' \tag{141}$$

By means of the former we can rewrite the latter as

$$(f_a - f_b)(\vec{p}_a - \vec{p}_a{}') = 0 \ . \tag{142}$$

That is, either our scattering process is in fact a trivial process where particles a and b do not interact—$\vec{p}_a{}' = \vec{p}_a$ and $\vec{p}_b{}' = \vec{p}_b$—or their gravitational couplings must be identical: $f_a = f_b \equiv f$. Notice that a and b are arbitrary particle species, and that there is no loss of generality in restricting to elastic processes: if two particles interact at all, as a consequence of the optical theorem the scattering amplitude for elastic processes cannot vanish. We have therefore shown that *all* particles must share the same gravitational coupling constant f.

This is nothing but the *equivalence principle*: at low energies, gravity couples to all forms of energy-momentum with the same strength, regardless of 'chemical composition'. Gravitational interactions are insensitive to the particle species—they only care about energy and momentum. We see here that the equivalence principle, rather than being an independent principle, it is in fact an unavoidable consequence of Lorentz invariance and quantum mechanics for massless spin-2 particles

There is one *caveat* for our proof that F^μ must be proportional to the total four-momentum. Suppose that particle species can be divided into two (or more) subsets A and B that do not

interact with each other. That is, particles belonging to one subset have non-trivial interactions within the same subset, but do not interact at all with particles belonging to the other. In such a case, in any scattering process we can decompose the initial and final states as

$$\alpha = \alpha_A + \alpha_B , \qquad \beta = \beta_A + \beta_B , \tag{143}$$

where $\alpha_{A,B}$ and $\beta_{A,B}$ are the initial and final states for the individual subsets. The scattering amplitude factorizes like in fig. 4, and besides total four-momentum conservation, we also have that the total four-momentum *within each subset* is conserved:

$$P^\mu_{\alpha_A} = P^\mu_{\beta_A} , \qquad P^\mu_{\alpha_B} = P^\mu_{\beta_B} . \tag{144}$$

As a consequence, we can split our new four-vector F^μ as $F^\mu = F^\mu_A + F^\mu_B$, and each of the two contributions must be separately conserved. According to our proof above, this is only possible if

$$F^\mu_A = f_A \, P^\mu_A , \qquad F^\mu_B = f_B \, P^\mu_B . \tag{145}$$

That is, subsystems of the world that do not interact with each other can have different gravitational coupling constants. But if f_A and f_B are both non-zero, both subsystems do interact with gravity, and therefore do interact with each other, at least through diagrams like fig. 5—in which case $P^\mu_{A,B}$ are *not* separately conserved, and we are back to the case $f_A = f_B = f$! The only other option is that one of the two subsystems, say B, does not interact with gravity either, $f_B = 0$. In a sense we have to decide which subsystem gravitons themselves belong to—and *we* want to belong to the same. The other subsystem—B—does not interact with us in any way, not even gravitationally. All particles that we know of—and that we can ever know of—are those in our subsystem. The existence of B cannot be probed, because to probe something you have to interact with it (as Heisenberg taught us).

One final remark: we never specified what kind of particles participate in the scattering $\alpha \to \beta$ as external states. Our proof was completely general: the particles in question may be massive or massless, with arbitrary spin. In particular, some of them may themselves be gravitons! Then, their coupling to a very soft graviton must be the same as for the other particles, $\propto f p^\mu p^\nu$. Therefore, not only low-energy gravitational interactions cannot distinguish between different 'ordinary' particles—they cannot tell the difference between a graviton and an ordinary particle either. This fact is sometimes referred to as the *strong equivalence principle*: as far as gravitational interactions are concerned, the energy and momentum stored in gravitational fields are indistinguishable from ordinary matter's energy and momentum. Once again, we see that in QFT this is not a principle, but a theorem.

We conclude this section by questioning the validity of our fundamental assumption, eq. (134). We can proceed in exactly the same way as for the photon—see last section. The most generic form of the elementary vertex amplitude (fig. 1) is

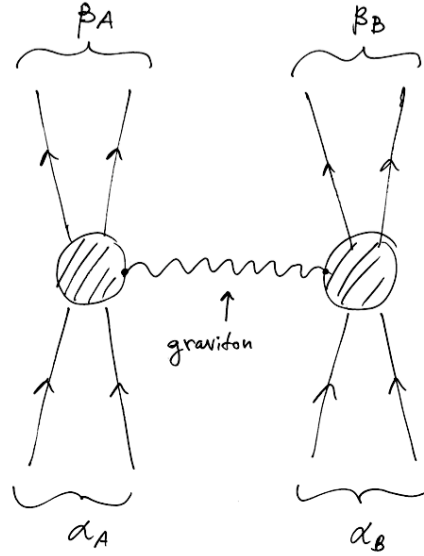$$\mathcal{M} = M^{\mu\nu}(p, \sigma, \sigma') \cdot e^*_{\mu\nu}(q) . \tag{146}$$

Figure 5: *A contribution to the A-B interaction, when both sectors interact with gravity.*

Plugging this into the amplitude for fig. 2b, and neglecting the contributions from attaching the graviton to internal particle lines, we get

$$
\text{fig. 2b} = \sum_n \sum_{\sigma'_n} \mathcal{M}_{\alpha\beta}(\dots, \sigma'_n, \dots) \frac{\eta_n \, M_n^{\mu\nu}(p_n, \sigma_n, \sigma'_n)}{p_n \cdot q} \, e^*_{\mu\nu}(q) \; , \tag{147}
$$

where like in the photon case, $\mathcal{M}_{\alpha\beta}(\dots, \sigma'_n, \dots)$ is the amplitude for the original process $\alpha \to \beta$ with the n-th spin replaced by σ'_n. For Lorentz invariance to hold we must have

$$
\sum_n \sum_{\sigma'_n} \mathcal{M}_{\alpha\beta}(\dots, \sigma'_n, \dots) \frac{\eta_n \, M_n^{\mu\nu}(p_n, \sigma_n, \sigma'_n) \, q_\mu}{p_n \cdot q} = 0 \; . \tag{148}
$$

Extrapolating this relation to generic q^μ's (i.e., with $q^2 \neq 0$), for any given n we can take q^μ to be orthogonal to p_n^μ and not orthogonal to any other momentum. The n-th denominator thus vanishes, and the resulting divergence can only be canceled if the n-th numerator also vanishes:

$$
M_n^{\mu\nu}(p_n, \sigma_n, \sigma'_n) \, q_\mu = 0 \qquad \forall q^\mu \text{ such that } q \cdot p_n = 0 \; , \tag{149}
$$

This is only possible if

$$
M_n^{\mu\nu}(p_n, \sigma_n, \sigma'_n) = p_n^\mu \, p_n^\nu \times f_n(p_n, \sigma_n, \sigma'_n) \; , \tag{150}
$$

where we used that $M_n^{\mu\nu}$ has to be symmetric. The above structure for $M_n^{\mu\nu}$ resembles eq. (134), with a momentum- and spin-dependent coupling constant f_n. Our constraint (148) becomes

$$
\sum_n \sum_{\sigma'_n} \mathcal{M}_{\alpha\beta}(\dots, \sigma'_n, \dots) \times \eta_n \, f_n(p_n, \sigma_n, \sigma'_n) \, p_n^\mu = 0 \; . \tag{151}
$$

This equation, rather than constraining the graviton couplings, in general imposes a linear relation between different scattering amplitudes in the original system, *before* taking into account the coupling with gravity—relation that depends on how eventually the various particles are going to couple to gravity. At least at lowest order in the gravitational coupling constants, this makes no sense. The only way out is that the gravitational couplings are diagonal in spin-space,

$$f_n(p_n, \sigma_n, \sigma'_n) = f_n(p_n, \sigma_n) \times \delta_{\sigma_n \sigma'_n} \ , \tag{152}$$

so that we get an overall $\mathcal{M}_{\alpha\beta}$ out of the sum over external particles, and we are left with

$$\mathcal{M}_{\alpha\beta} \sum_n \eta_n \, f_n(p_n, \sigma_n) \, p_n^\mu = 0 \ . \tag{153}$$

Either the original amplitude $\mathcal{M}_{\alpha\beta}$ vanishes, or the four vector

$$F^\mu = \sum f_n(p_n, \sigma_n) \, p_n^\mu \tag{154}$$

has to be conserved by our scattering process. But, as we showed above, the only additive four-vector that is conserved by all non-trivial scattering processes is the total four-momentum. Therefore F^μ must be proportional to the total four-momentum, which means that the gravitational coupling constants are independent of momenta and spins, as well as of the particle species:

$$f_n(p_n, \sigma_n) = f \qquad \forall n \ . \tag{155}$$

This completes our proof.

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