One feature of the magnetic force law (Eq. 5.1) warrants special attention:

$$\boxed{\textbf{Magnetic forces do no work.}}$$

For if Q moves an amount $d\mathbf{l} = \mathbf{v}\, dt$, the work done is

$$dW_{\text{mag}} = \mathbf{F}_{\text{mag}} \cdot d\mathbf{l} = Q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}\, dt = 0. \tag{5.11}$$

This follows because $(\mathbf{v} \times \mathbf{B})$ is perpendicular to $\mathbf{v}$, so $(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} = 0$. Magnetic forces may alter the *direction* in which a particle moves, but they cannot speed it up or slow it down. The fact that magnetic forces do no work is an elementary and direct consequence of the Lorentz force law, but there are many situations in which it *appears* so manifestly false that one's confidence is bound to waver. When a magnetic crane lifts the carcass of a junked car, for instance, *something* is obviously doing work, and it seems perverse to deny that the magnetic force is responsible. Well, perverse or not, deny it we must, and it can be a very subtle matter to figure out what agency *does* deserve the credit in such circumstances. I'll show you several examples as we go along.

Problem 5.1 A particle of charge q enters a region of uniform magnetic field $\mathbf{B}$ (pointing *into* the page). The field deflects the particle a distance d above the original line of flight, as shown in Fig. 5.8. Is the charge positive or negative? In terms of a, d, B and q, find the momentum of the particle.

Problem 5.2 Find and sketch the trajectory of the particle in Ex. 5.2, if it starts at the origin with velocity

(*a*) $\mathbf{v}(0) = (E/B)\hat{\mathbf{y}}$,

(*b*) $\mathbf{v}(0) = (E/2B)\hat{\mathbf{y}}$,

(*c*) $\mathbf{v}(0) = (E/B)(\hat{\mathbf{y}} + \hat{\mathbf{z}})$.

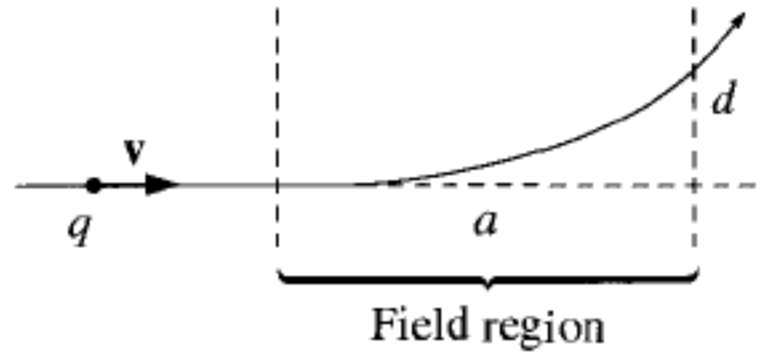

Figure 5.8

Problem 5.3 In 1897 J. J. Thomson "discovered" the electron by measuring the charge-to-mass ratio of "cathode rays" (actually, streams of electrons, with charge q and mass m) as follows:

(a) First he passed the beam through uniform crossed electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ (mutually perpendicular, and both of them perpendicular to the beam), and adjusted the electric field until he got zero deflection. What, then, was the speed of the particles (in terms of E and B)?

(b) Then he turned off the electric field, and measured the radius of curvature, R, of the beam, as deflected by the magnetic field alone. In terms of E, B, and R, what is the charge-to-mass ratio (q/m) of the particles?

5.1.3 Currents

The **current** in a wire is the *charge per unit time* passing a given point. By definition, negative charges moving to the left count the same as positive ones to the right. This conveniently reflects the *physical* fact that almost all phenomena involving moving charges depend on the *product* of charge and velocity—if you change the sign of q *and* $\mathbf{v}$, you get the same answer, so it doesn't really matter which you have. (The Lorentz force law is a case in point; the Hall effect (Prob. 5.39) is a notorious exception.) In practice, it is ordinarily the negatively charged electrons that do the moving—in the direction *opposite* the electric current. To avoid the petty complications this entails, I shall often pretend it's the positive charges that move, as in fact everyone assumed they did for a century or so after Benjamin Franklin established his unfortunate convention.[3] Current is measured in coulombs-per-second, or **amperes** (A):

$$1 \text{ A} = 1 \text{ C/s}. \tag{5.12}$$

A line charge λ traveling down a wire at speed v (Fig. 5.9) constitutes a current

$$I = \lambda v, \tag{5.13}$$

because a segment of length $v\Delta t$, carrying charge $\lambda v \Delta t$, passes point P in a time interval Δt. Current is actually a *vector:*

$$\mathbf{I} = \lambda \mathbf{v}; \tag{5.14}$$

[3] If we called the electron plus and the proton minus, the problem would never arise. In the context of Franklin's experiments with cat's fur and glass rods, the choice was completely arbitrary.

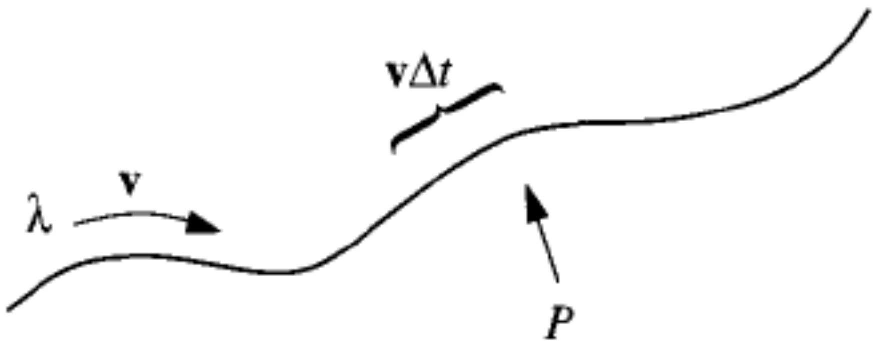

Figure 5.9

since the path of the flow is dictated by the shape of the wire, most people don't bother to display the vectorial character of $\mathbf{I}$ explicitly, but when it comes to surface and volume currents we cannot afford to be so casual, and for the sake of notational consistency it is a good idea to acknowledge this right from the start. A neutral wire, of course, contains as many stationary positive charges as mobile negative ones. The former do not contribute to the current—the charge density λ in Eq. 5.13 refers only to the *moving* charges. In the unusual situation where *both* types move, $\mathbf{I} = \lambda_+\mathbf{v}_+ + \lambda_-\mathbf{v}_-$.

The magnetic force on a segment of current-carrying wire is evidently

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\, dq = \int (\mathbf{v} \times \mathbf{B})\lambda\, dl = \int (\mathbf{I} \times \mathbf{B})\, dl. \tag{5.15}$$

Inasmuch as $\mathbf{I}$ and $d\mathbf{l}$ both point in the same direction, we can just as well write this as

$$\boxed{\mathbf{F}_{\text{mag}} = \int I\, (d\mathbf{l} \times \mathbf{B}).} \tag{5.16}$$

Typically, the current is constant (in magnitude) along the wire, and in that case I comes outside the integral:

$$\mathbf{F}_{\text{mag}} = I \int (d\mathbf{l} \times \mathbf{B}). \tag{5.17}$$

Example 5.3

A rectangular loop of wire, supporting a mass m, hangs vertically with one end in a uniform magnetic field $\mathbf{B}$, which points into the page in the shaded region of Fig. 5.10. For what current I, in the loop, would the magnetic force upward exactly balance the gravitational force downward?

Solution: First of all, the current must circulate clockwise, in order for $(\mathbf{I} \times \mathbf{B})$ in the horizontal segment to point upward. The force is

$$F_{\text{mag}} = IBa,$$

where a is the width of the loop. (The magnetic forces on the two vertical segments cancel.) For F_{mag} to balance the weight (mg), we must therefore have

$$I = \frac{mg}{Ba}. \tag{5.18}$$

The weight just *hangs* there, suspended in mid-air!

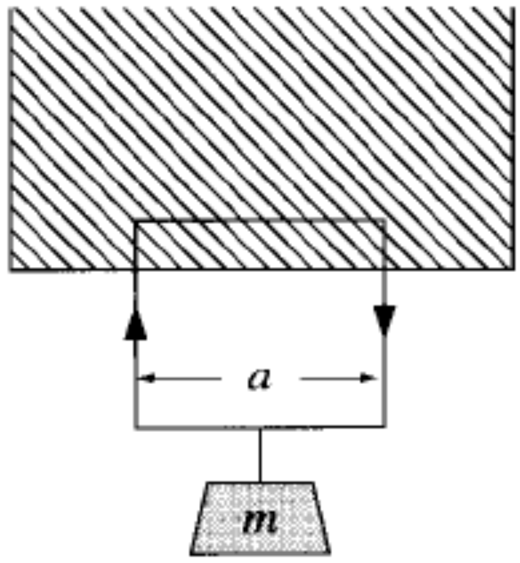

Figure 5.10

What happens if we now *increase* the current? Then the upward magnetic force *exceeds* the downward force of gravity, and the loop rises, lifting the weight. *Somebody's* doing work, and it sure looks as though the magnetic force is responsible. Indeed, one is tempted to write

$$W_{\text{mag}} = F_{\text{mag}}h = IBah, \tag{5.19}$$

where h is the distance the loop rises. But we know that magnetic forces *never* do work. What's going on here?

Well, when the loop starts to rise, the charges in the wire are no longer moving horizontally—their velocity now acquires an upward component u, the speed of the loop (Fig. 5.11), in addition to the horizontal component w associated with the current ($I = \lambda w$). The magnetic force, which is always perpendicular to the velocity, no longer points straight up, but tilts back. It is perpendicular to the *net* displacement of the charge (which is in the direction of $\mathbf{v}$), and therefore *it does no work on* q. It does have a vertical *component* (qwB); indeed, the net vertical force on all the charge (λa) in the upper segment of the loop is

$$F_{\text{vert}} = \lambda awB = IBa \tag{5.20}$$

(as before); but now it also has a *horizontal* component (quB), which opposes the flow of current. Whoever is in charge of maintaining that current, therefore, must now *push* those charges along, against the backward component of the magnetic force.

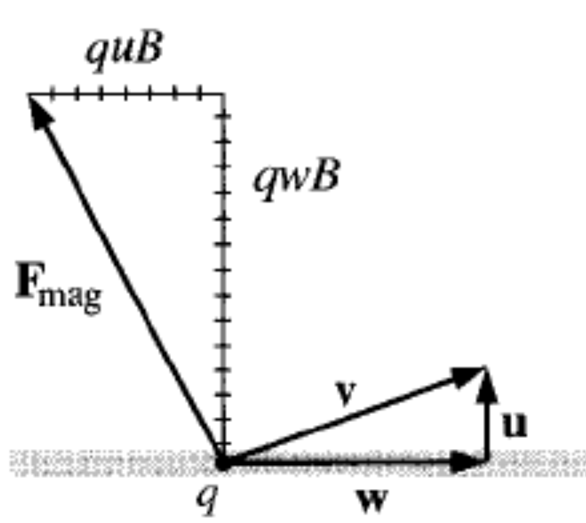

Figure 5.11

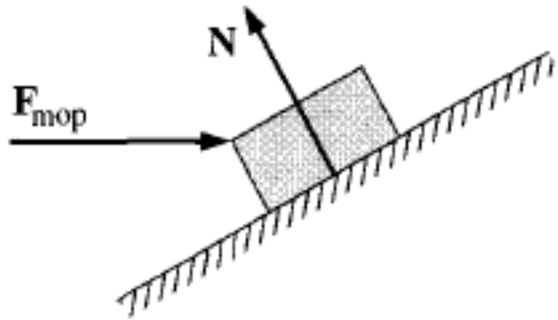

Figure 5.12

The total horizontal force on the top segment is evidently

$$F_{\text{horiz}} = \lambda a u B. \tag{5.21}$$

In a time dt the charges move a (horizontal) distance $w\,dt$, so the work done by this agency (presumably a battery or a generator) is

$$W_{\text{battery}} = \lambda a B \int u w \, dt = I B a h,$$

which is precisely what we naïvely attributed to the *magnetic* force in Eq. 5.19. Was work done in this process? Absolutely! Who *did* it? The battery! What, then, was the role of the magnetic force? Well, it *redirected* the horizontal force of the battery into the *vertical* motion of the loop and the weight.