We now consider the propagation of a wave packet in a dispersive medium. The time development of the wave packet is given by the integral over k in Eq. (2). The angular frequency ω in the integral is a function of k given by

$$\omega(k) = \frac{ck}{n(k)}, \tag{6}$$

where we have considered the index refraction n as a function of k. Then, the integral in Eq. (2) is

$$f(x,t) = \int_{-\infty}^{+\infty} dk A(k) e^{i[kx-\omega(k)t]}. \tag{7}$$

If the medium is non-dispersive, n is a constant and the integral becomes

$$f(x,t) = \int_{-\infty}^{+\infty} dk A(k) e^{ik[x-(c/n)t]}. \tag{8}$$

Comparing this integral with that in Eq. (2), for $t = 0$ in that integral, we see that Eq. (8) results in

$$f(x,t) = f\left[k\left(x - \frac{c}{n}t\right), 0\right]. \tag{9}$$

This shows that, in a non-dispersive medium, the wave packet retains its original shape, while moving with velocity $v = c/n$.

In a dispersive medium, we expand the function $\omega(k)$ about the value k_0 at which the distribution in k of the wave packet peaks:

$$\omega(k) = \omega_0 + (k - k_0)\omega_0' + ..., \tag{10}$$

where $\omega_0 = \omega(k_0)$ and $\omega_0' = \frac{d\omega}{dk}|_{k_0}$. We have expanded $\omega(k)$ up to first order in $(k - k_0)$. With this approximation, the wave packet integral can be written as

$$f(x,t) = e^{-i(\omega_0 - k_0\omega_0')t} \int_{-\infty}^{+\infty} dk\ A(k) e^{ik(x-\omega_0' t)} \tag{11}$$

Comparison of this with Eqs. (2) and (4) now gives

$$\begin{aligned} f(x,t) &= e^{-i(\omega_0 - k_0\omega_0')t} f\left[(x - \omega_0' t), 0\right] \\ &= g(x - \omega_0' t) e^{i(k_0 x - \omega_0 t)}. \end{aligned} \tag{12}$$

This shows that, in a dispersive medium, the envelope of a wave packet will retain its original shape [to first order in the expansion of $\omega(k)$], and move with a **group velocity**

$$v_g = \omega_0' = \frac{d\omega}{dk}, \tag{13}$$

with the understanding that the derivative is taken at a central wave number k_0. The velocity at which the waves within the packet move, called the **phase velocity**, is given by

$$v_p = \frac{\omega_0}{k_0} = \frac{c}{n}. \tag{14}$$