chapter 1

PRELIMINARIES

1.1 LINEAR ALGEBRA

We assume that the reader is familiar with the contents of a standard course in linear algebra, including finite-dimensional vector spaces, subspaces, linear transformations and matrices, determinants, eigenvalues, bilinear and quadratic forms, positive definiteness, inner product spaces, and orthogonal linear transformations. Accounts of these topics may be found in most linear algebra books (e.g., [14] or [21]). Throughout the book V will denote a real Euclidean vector space, i.e., a finite-dimensional inner product space over the real field $\mathscr{R}$. Partly in order to establish notation we list some of the properties of V that are of importance for the ensuing discussion.

If X and Y are subsets of V such that $(x, y) = 0$ for all $x \in X$ and all $y \in Y$, we shall say that X and Y are *orthogonal*, or *perpendicular*, and write $X \perp Y$. If $X \subseteq V$, the *orthogonal complement* of X, which is the subspace of V consisting of all $x \in V$ such that $x \perp X$, will be denoted by $X^{\perp}$. If W is a subspace of V, then $W^{\perp\perp} = W$ and $V = W \oplus W^{\perp}$.

If $\{x_1, \ldots, x_n\}$ is a basis for V, let V_i be the subspace spanned by $\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}$, excluding x_i. If $0 \neq y_i \in V_i^{\perp}$, then $(x_j, y_i) = 0$ for all $j \neq i$, but $(x_i, y_i) \neq 0$, for otherwise $y_i \in V^{\perp} = 0$. Dividing y_i by (x_i, y_i), if necessary, we may assume that $(x_i, y_i) = 1$, thereby making y_i unique since $\dim(V_i^{\perp}) = 1$. Observe that if $\Sigma_{i=1}^{n} \lambda_i y_i = 0$ with $\lambda_i \in \mathscr{R}$, then

$$0 = (x_j, 0) = (x_j, \Sigma_i \lambda_i y_i) = \Sigma_i \lambda_i (x_j, y_i) = \lambda_j$$

for all j, and so $\{y_1, \ldots, y_n\}$ is linearly independent. Thus $\{y_1, \ldots, y_n\}$ is a