

## chapter 2

# FINITE GROUPS IN TWO AND THREE DIMENSIONS

### 2.1 ORTHOGONAL TRANSFORMATIONS IN TWO DIMENSIONS

If  $T \in \mathcal{O}(\mathbb{R}^2)$ , then  $T$  is completely determined by its action on the basis vectors  $e_1 = (1, 0)$  and  $e_2 = (0, 1)$ . If  $Te_1 = (\mu, \nu)$ , then  $\mu^2 + \nu^2 = 1$  and  $Te_2 = \pm(-\nu, \mu)$ , since  $T$  preserves length and orthogonality. Choose  $\theta, 0 \leq \theta < 2\pi$ , such that  $\cos \theta = \mu$  and  $\sin \theta = \nu$ .

If  $Te_2 = (-\nu, \mu)$ , then  $T$  is represented by the matrix

$$A = \begin{bmatrix} \mu & -\nu \\ \nu & \mu \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix},$$

and it is clear that  $T$  is a counterclockwise rotation of the plane about the origin through the angle  $\theta$  (see Figure 2.1). Observe that

$$\det T = \mu^2 + \nu^2 = \cos^2 \theta + \sin^2 \theta = 1.$$

If  $Te_2 = (\nu, -\mu)$ , then  $T$  is represented by the matrix

$$B = \begin{bmatrix} \mu & \nu \\ \nu & -\mu \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}.$$

In this case observe that

$$\det T = -\cos^2 \theta - \sin^2 \theta = -1,$$

and that

$$B^2 = \begin{bmatrix} \mu^2 + \nu^2 & 0 \\ 0 & \mu^2 + \nu^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

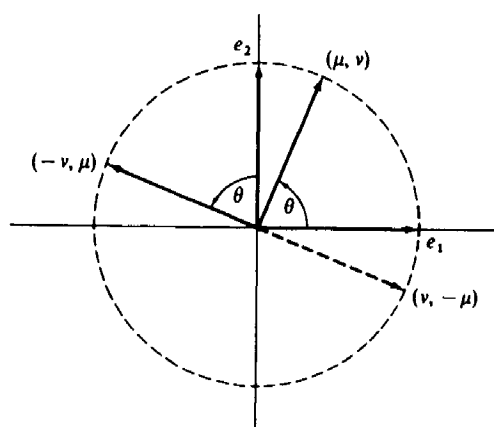


Figure 2.1

so that  $T^2 = 1$ . It is easy to verify (Exercise 2.1) that the vector  $x_1 = (\cos \theta/2, \sin \theta/2)$  is an eigenvector having eigenvalue 1 for  $T$ , so that the line  $l = \{\lambda x_1 : \lambda \in \mathcal{R}\}$  is left pointwise fixed by  $T$ . Similarly, the vector  $x_2 = (-\sin \theta/2, \cos \theta/2)$  is an eigenvector with eigenvalue  $-1$ , and  $x_2 \perp x_1$  [see Figure 2.2(a)]. With respect to the basis  $\{x_1, x_2\}$  the transformation  $T$  is represented by the matrix

$$C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

If  $x = \lambda_1 x_1 + \lambda_2 x_2$ , then  $Tx = \lambda_1 x_1 - \lambda_2 x_2$ , and  $T$  sends  $x$  to its mirror

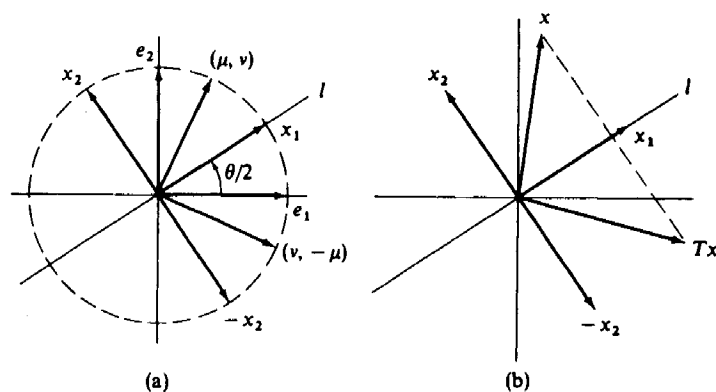


Figure 2.2

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## 2.2 FINITE

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