5 General Relativity with Tetrads

5.1 Concept Questions

1. The vierbein has 16 degrees of freedom instead of the 10 degrees of freedom of the metric. What do the extra 6 degrees of freedom correspond to?

2. Tetrad transformations are defined to be Lorentz transformations. Don't general coordinate transformations already include Lorentz transformations as a particular case, so aren't tetrad transformations redundant?

3. What does coordinate gauge-invariant mean? What does tetrad gauge-invariant mean?

4. Is the coordinate metric $g_{\mu\nu}$ tetrad gauge-invariant?

5. What does a directed derivative ∂_m mean physically?

6. Is the directed derivative ∂_m coordinate gauge-invariant?

7. What is the tetrad-frame 4-velocity u^m of a person at rest in an orthonormal tetrad frame?

8. If the tetrad frame is accelerating (not in free-fall) does the 4-velocity u^m of a person continuously at rest in the tetrad frame change with time? Is it true that $\partial_t u^m = 0$? Is it true that $D_t u^m = 0$?

9. If the tetrad frame is accelerating, do the tetrad axes $\boldsymbol{\gamma}_m$ change with time? Is it true that $\partial_t \boldsymbol{\gamma}_m = 0$? Is it true that $D_t \boldsymbol{\gamma}_m = 0$?

10. If an observer is accelerating, do the observer's locally inertial rest axes $\boldsymbol{\gamma}_m$ change along the observer's wordline? Is it true that $\partial_t \boldsymbol{\gamma}_m = 0$? Is it true that $D_t \boldsymbol{\gamma}_m = 0$?

11. If the tetrad frame is accelerating, does the tetrad metric γ_{mn} change with time? Is it true that $\partial_t \gamma_{mn} = 0$? Is it true that $D_t \gamma_{mn} = 0$?

12. If the tetrad frame is accelerating, do the covariant components u_m of the 4-velocity of a person continuously at rest in the tetrad frame change with time? Is it true that $\partial_t u_m = 0$? Is it true that $D_t u_m = 0$?

13. Suppose that $\boldsymbol{p} = \boldsymbol{\gamma}_m p^m$ is a 4-vector. Is the proper rate of change of the proper components p^m measured by an observer equal to the directed time derivative $\partial_t p^m$ or to the covariant time derivative $D_t p^m$? What about the covariant components p_m of the 4-vector? [Hint: The proper contravariant components of the 4-vector measured by an observer are $p^m \equiv \boldsymbol{\gamma}^m \cdot \boldsymbol{p}$ where $\boldsymbol{\gamma}^m$ are the contravariant locally inertial rest axes of the observer. Similarly the proper covariant components are $p_m \equiv \boldsymbol{\gamma}_m \cdot \boldsymbol{p}$.]

14. A person with two eyes separated by proper distance $\delta\xi^n$ observes an object. The observer observes the photon 4-vector from the object to be p^m. The observer uses the difference δp^m in the two 4-vectors detected by the two eyes to infer the binocular distance to the object. Is the difference δp^m in photon 4-vectors detected by the two eyes equal to the directed derivative $\delta\xi^n \partial_n p^m$ or to the covariant derivative $\delta\xi^n D_n p^m$?

15. What does parallel-transport mean?

16. Suppose that p^m is a tetrad 4-vector. Parallel-transport the 4-vector by an infinitesimal proper distance $\delta\xi^n$. Is the change in p^m measured by an ensemble of observers at rest in the tetrad frame equal to the directed derivative $\delta\xi^n \partial_n p^m$ or to the covariant derivative $\delta\xi^n D_n p^m$? [Hint: What if "rest" means that the observer at each point is separately at rest in the tetrad frame at that point? What if "rest" means that the observers are mutually at rest relative to each other in the rest frame of the tetrad at one particular point?]

17. What is the physical significance of the fact that directed derivatives fail to commute?

18. Physically, what do the tetrad connection coefficients Γ_{kmn} mean?

19. What is the physical significance of the fact that Γ_{kmn} is antisymmetric in its first two indices (if the tetrad metric γ_{mn} is constant)?

20. Are the tetrad connections Γ_{kmn} coordinate gauge-invariant?

21. Explain how the equation for the Gullstrand-Painlevé metric in Cartesian coordinates $x^\mu \equiv \{t_{\rm ff}, x, y, z\}$
$$ds^2 = dt_{\rm ff}^2 - \delta_{ij}(dx^i - \beta^i dt_{\rm ff})(dx^j - \beta^j dt_{\rm ff}) \tag{1}$$
encodes not merely a metric but a full vierbein.

22. In what sense does the Gullstrand-Painlevé metric (1) depict a flow of space? [Are the coordinates moving? If not, then what is moving?]

23. If space has no substance, what does it mean that space falls into a black hole?

24. Would there be any gravitational field in a spacetime where space fell at constant velocity instead of accelerating?

25. In spherically symmetric spacetimes, what is the most important Einstein equation, the one that causes Reissner-Nordström black holes to be repulsive in their interiors, and causes mass inflation in non-empty (non Reissner-Nordström) charged black holes?

5.2 What's important?

This section of the notes describes the tetrad formalism of GR.

1. Why tetrads? Because physics is clearer in a locally inertial frame than in a coordinate frame.

2. The primitive object in the tetrad formalism is the vierbein ${e_m}^\mu$, in place of the metric in the coordinate formalism.

3. Written suitably, for example as equation (1), a metric ds^2 encodes not only the metric coefficients $g_{\mu\nu}$, but a full (inverse) vierbein ${e^m}_\mu$, through $ds^2 = \gamma_{mn} \, {e^m}_\mu dx^\mu \, {e^n}_\nu dx^\nu$.

4. The tetrad road from vierbein to energy-momentum is similar to the coordinate road from metric to energy-momentum, albeit a little more complicated.

5. In the tetrad formalism, the directed derivative ∂_m is the analog of the coordinate partial derivative $\partial/\partial x^\mu$ of the coordinate formalism. Directed derivatives ∂_m do not commute, whereas coordinate derivatives $\partial/\partial x^\mu$ do commute.

5.3 Tetrad

A **tetrad** (Greek foursome) $\boldsymbol{\gamma}_m(x)$ is a set of axes

$$\boxed{\boldsymbol{\gamma}_m \equiv \{\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3\}} \tag{2}$$

attached to each point x^μ of spacetime. The common case is that of an **orthonormal tetrad**, where the axes form a locally inertial frame at each point, so that the scalar products of the axes constitute the Minkowski metric η_{mn}

$$\boldsymbol{\gamma}_m \cdot \boldsymbol{\gamma}_n = \eta_{mn} \ . \tag{3}$$

However, other tetrads prove useful in appropriate circumstances. There are spinor tetrads, null tetrads (notably the Newman-Penrose double null tetrad), and others (indeed, the basis of coordinate tangent vectors $\boldsymbol{g}_\mu$ is itself a tetrad). In general, the tetrad metric is some symmetric matrix γ_{mn}

$$\boxed{\boldsymbol{\gamma}_m \cdot \boldsymbol{\gamma}_n \equiv \gamma_{mn}} \ . \tag{4}$$

Associated with the tetrad frame at each point is a local set of coordinates

$$\xi^m \equiv \{\xi^0, \xi^1, \xi^2, \xi^3\} \ . \tag{5}$$

Unlike the coordinates x^μ of the background geometry, the local coordinates ξ^m do not extend beyond the local frame at each point. A coordinate interval is

$$\boldsymbol{dx} = \boldsymbol{\gamma}_m \, d\xi^m \tag{6}$$

and the scalar spacetime distance is

$$ds^2 = \boldsymbol{dx} \cdot \boldsymbol{dx} = \gamma_{mn} \, d\xi^m d\xi^n \ . \tag{7}$$

Andrew's convention:
Latin dummy indices label tetrad frames.
Greek dummy indices label coordinate frames.

Why introduce tetrads?

1. The physics is more transparent when expressed in a locally inertial frame (or some other frame adapted to the physics), as opposed to the coordinate frame, where Salvador Dali rules.
2. If you want to consider spin-$\frac{1}{2}$ particles and quantum physics, you better work with tetrads.
3. For good reason, much of the GR literature works with tetrads, so it's useful to understand them.

5.4 Vierbein

The **vierbein** (German four-legs) $e_m{}^\mu$ is defined to be the matrix that transforms between the tetrad frame and the coordinate frame (note the placement of indices: the tetrad index m comes first, then the coordinate index μ)

$$\boxed{\boldsymbol{\gamma}_m = e_m{}^\mu \, \boldsymbol{g}_\mu} \; . \tag{8}$$

The vierbein is a 4×4 matrix, with 16 independent components. The inverse vierbein $e^m{}_\mu$ is defined to be the matrix inverse of the vierbein $e_m{}^\mu$, so that

$$e^m{}_\mu \, e_m{}^\nu = \delta^\nu_\mu \; , \quad e^m{}_\mu \, e_n{}^\mu = \delta^n_m \; . \tag{9}$$

Thus equation (8) inverts to

$$\boxed{\boldsymbol{g}_\mu = e^m{}_\mu \, \boldsymbol{\gamma}_m} \; . \tag{10}$$

5.5 The metric encodes the vierbein

The scalar spacetime distance is

$$ds^2 = \gamma_{mn} \, e^m{}_\mu dx^\mu \, e^n{}_\nu dx^\nu = g_{\mu\nu} \, dx^\mu \, dx^\nu \tag{11}$$

from which it follows that the coordinate metric $g_{\mu\nu}$ is

$$\boxed{g_{\mu\nu} = \gamma_{mn} \, e^m{}_\mu \, e^n{}_\nu} \; . \tag{12}$$

The shorthand way in which metric's are commonly written encodes not only a metric but also an inverse vierbein, hence a tetrad. For example, the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 \tag{13}$$

encodes the inverse vierbein

$$e^t{}_\mu dx^\mu = \left(1 - \frac{2M}{r}\right)^{1/2} dt \; , \tag{14a}$$

$$e^r{}_\mu dx^\mu = \left(1 - \frac{2M}{r}\right)^{-1/2} dr \; , \tag{14b}$$

$$e^\theta{}_\mu dx^\mu = r \, d\theta \; , \tag{14c}$$

$$e^\phi{}_\mu dx^\mu = r \sin\theta \, d\phi \; , \tag{14d}$$

Explicitly, the inverse vierbein of the Schwarzschild metric is is the diagonal matrix

$$e^m{}_\mu = \begin{pmatrix} (1 - 2M/r)^{1/2} & 0 & 0 & 0 \\ 0 & (1 - 2M/r)^{-1/2} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r \sin\theta \end{pmatrix} \; . \tag{15}$$

5.6 Tetrad transformations

Tetrad transformations are defined to be Lorentz transformations. The Lorentz transformation may be a different transformation at each point. Tetrad transformations rotate the tetrad axes γ_k at each point by a Lorentz transformation $L_k{}^m$, while keeping the background coordinates x^μ unchanged:

$$\boxed{\boldsymbol{\gamma}_k \rightarrow \boldsymbol{\gamma}'_k = L_k{}^m \, \boldsymbol{\gamma}_m} \ . \tag{16}$$

In the case that the tetrad axes γ_k are orthonormal, with a Minkowski metric, the Lorentz transformation matrices $L_k{}^m$ in equation (16) take the familiar special relativistic form, but the linear matrices $L_k{}^m$ in equation (16) signify a Lorentz transformation in any case.

Whether or not the tetrad axes are orthonormal, Lorentz transformations are precisely those transformations that leave the tetrad metric unchanged

$$\gamma'_{kl} = \boldsymbol{\gamma}'_k \cdot \boldsymbol{\gamma}'_l = L_k{}^m L_l{}^n \, \boldsymbol{\gamma}_m \cdot \boldsymbol{\gamma}_n = L_k{}^m L_l{}^n \, \gamma_{mn} = \gamma_{kl} \ . \tag{17}$$

5.7 Tetrad Tensor

In general, a tetrad-frame **tensor** $A^{kl...}_{mn...}$ is an object that transforms under tetrad (Lorentz) transformations (16) as

$$\boxed{A'^{kl...}_{mn...} = L^k{}_a L^l{}_b \ldots L_m{}^c L_n{}^d \ldots A^{ab...}_{cd...}} \ . \tag{18}$$

5.8 Raising and lowering indices

In the coordinate approach to GR, coordinate indices were lowered and raised with the coordinate metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$. In the tetrad formalism there are two kinds of indices, tetrad indices and coordinate indices, and they flip around as follows:

1. Lower and raise coordinate indices with the coordinate metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$;
2. Lower and raise tetrad indices with the tetrad metric γ_{mn} and its inverse γ^{mn};
3. Switch between coordinate and tetrad frames with the vierbein $e_m{}^\mu$ and its inverse $e^m{}_\mu$.

The kinds of objects for which this flippery is valid are called tensors. Tensors with only tetrad indices, such as the tetrad axes $\boldsymbol{\gamma}_m$ or the tetrad metric γ_{mn} are called tetrad tensors, and they remain unchanged under coordinate transformations. Tensors with only coordinate indices, such as the coordinate tangent axes $\boldsymbol{g}_\mu$ or the coordinate metric $g_{\mu\nu}$, are called coordinate tensors, and they remain unchanged under tetrad transformations. Tensors may also be mixed, such as the vierbein $e_m{}^\mu$.

5.9 Gauge transformations

Gauge transformations are transformations of the coordinates or tetrad. Such transformations do not change the underlying spacetime.

Quantities that are unchanged by a coordinate transformation are **coordinate gauge-invariant**. Quantities that are unchanged under a tetrad transformation are **tetrad gauge-invariant**. For example, tetrad tensors are coordinate gauge-invariant, while coordinate tensors are tetrad gauge-invariant.

Tetrad transformations have the 6 degrees of freedom of Lorentz transformations, with 3 degrees of freedom in spatial rotations, and 3 more in Lorentz boosts. General coordinate transformations have 4 degrees of freedom. Thus there are 10 degrees of freedom in the choice of tetrad and coordinate system. The 16 degrees of freedom of the vierbein, minus the 10 degrees of freedom from the transformations of the tetrad and coordinates, leave 6 physical degrees of freedom in spacetime, the same as in the coordinate approach to GR, which is as it should be.

5.10 Directed derivatives

Directed derivatives ∂_m are defined to be the directional derivatives along the axes $\boldsymbol{\gamma}_m$

$$\boxed{\partial_m \equiv \boldsymbol{\gamma}_m \cdot \boldsymbol{\partial} = \boldsymbol{\gamma}_m \cdot \boldsymbol{g}^\mu \frac{\partial}{\partial x^\mu} = e_m{}^\mu \frac{\partial}{\partial x^\mu}} \quad \textit{is} \text{ a tetrad-frame 4-vector .} \tag{19}$$

The directed derivative ∂_m is independent of the choice of coordinates, as signaled by the fact that it has only a tetrad index, no coordinate index.

Unlike coordinate derivatives $\partial/\partial x^\mu$, directed derivatives ∂_m do not commute. Their commutator is

$$\begin{aligned}
[\partial_m, \partial_n] &= \left[e_m{}^\mu \frac{\partial}{\partial x^\mu} \, , \, e_n{}^\nu \frac{\partial}{\partial x^\nu} \right] \\
&= e_m{}^\mu \frac{\partial e_n{}^\nu}{\partial x^\mu} \frac{\partial}{\partial x^\nu} - e_n{}^\nu \frac{\partial e_m{}^\mu}{\partial x^\nu} \frac{\partial}{\partial x^\mu} \\
&= (d^k_{nm} - d^k_{mn}) \, \partial_k \quad \text{is } \textit{not} \text{ a tensor}
\end{aligned} \tag{20}$$

where $d_{lmn} \equiv \gamma_{lk} \, d^k_{mn}$ is the vierbein derivative

$$\boxed{d_{lmn} \equiv \gamma_{lk} \, e^k{}_\kappa \, e_n{}^\nu \frac{\partial e_m{}^\kappa}{\partial x^\nu}} \quad \text{is } \textit{not} \text{ a tensor .} \tag{21}$$

Since the vierbein and inverse vierbein are inverse to each other, an equivalent definition of d_{lmn} in terms of the inverse vierbein is

$$d_{lmn} \equiv \; - \gamma_{lk} \, e_m{}^\mu \, e_n{}^\nu \frac{\partial e^k{}_\mu}{\partial x^\nu} \quad \text{is } \textit{not} \text{ a tensor .} \tag{22}$$

5.11 Tetrad covariant derivative

The derivation of tetrad covariant derivatives D_m follows precisely the analogous derivation of coordinate covariant derivatives D_μ. The tetrad-frame formulae look entirely similar to the coordinate-frame formulae, with the replacement of coordinate partial derivatives by directed derivatives, $\partial/\partial x^\mu \to \partial_m$, and the replacement of coordinate-frame connections by tetrad-frame connections $\Gamma^\kappa_{\mu\nu} \to \Gamma^k_{mn}$. There are two things to be careful about: first, unlike coordinate partial derivatives, directed derivatives ∂_m do not commute; and second, neither tetrad-frame nor coordinate-frame connections are tensors, and therefore it should be no surprise that the tetrad-frame connections Γ_{lmn} are *not* related to the coordinate-frame connections $\Gamma_{\lambda\mu\nu}$ by the 'usual' vierbein transformations. Rather, the tetrad and coordinate connections are related by equation (32).

If Φ is a scalar, then $\partial_m \Phi$ is a tetrad 4-vector. The tetrad covariant derivative of a scalar is just the directed derivative

$$\boxed{D_m \Phi = \partial_m \Phi \quad \textit{is a 4-vector}} \ . \tag{23}$$

If A^m is a tetrad 4-vector, then $\partial_n A^m$ is *not* a tensor, and $\partial_n A_m$ is *not* a tensor. But the 4-vector $\boldsymbol{A} = \boldsymbol{\gamma}_m A^m$, being by construction invariant under both tetrad and coordinate transformations, is a scalar, and its directed derivative is therefore a 4-vector

$$\begin{aligned} \partial_n \boldsymbol{A} &= \partial_n(\boldsymbol{\gamma}_m A^m) \quad \textit{is a 4-vector} \\ &= \boldsymbol{\gamma}_m \partial_n A^m + (\partial_n \boldsymbol{\gamma}_m) A^m \\ &= \boldsymbol{\gamma}_m \partial_n A^m + \Gamma^k_{mn} \boldsymbol{\gamma}_k \, A^m \end{aligned} \tag{24}$$

where the **tetrad-frame connection coefficients**, Γ^k_{mn}, also known as Ricci rotation coefficients (or, in the context of Newman-Penrose tetrads, spin coefficients) are defined by

$$\boxed{\partial_n \boldsymbol{\gamma}_m \equiv \Gamma^k_{mn} \, \boldsymbol{\gamma}_k} \quad \text{is } \textit{not} \text{ a tensor} \ . \tag{25}$$

Equation (24) shows that

$$\partial_n \boldsymbol{A} = \boldsymbol{\gamma}_k (D_n A^k) \quad \textit{is a tensor} \tag{26}$$

where $D_n A^k$ is the covariant derivative of the contravariant 4-vector A^k

$$\boxed{D_n A^k \equiv \partial_n A^k + \Gamma^k_{mn} A^m} \quad \textit{is a tensor} \ . \tag{27}$$

Similarly,

$$\partial_n \boldsymbol{A} = \boldsymbol{\gamma}^k (D_n A_k) \tag{28}$$

where $D_n A_k$ is the covariant derivative of the covariant 4-vector A_k

$$\boxed{D_n A_k \equiv \partial_n A_k - \Gamma^m_{kn} A_m} \quad \textit{is a tensor} \ . \tag{29}$$

In general, the covariant derivative of a tensor is

$$\boxed{D_a A^{kl...}_{mn...} = \partial_a A^{kl...}_{mn...} + \Gamma^k_{ba} A^{bl...}_{mn...} + \Gamma^l_{ba} A^{kb...}_{mn...} + ... - \Gamma^b_{ma} A^{kl...}_{bn...} - \Gamma^b_{na} A^{kl...}_{mb...} - ...} \tag{30}$$

with a positive Γ term for each contravariant index, and a negative Γ term for each covariant index.

5.12 Relation between tetrad and coordinate connections

The relation between the tetrad connections Γ^k_{mn} and their coordinate counterparts $\Gamma^\kappa_{\mu\nu}$ follows from

$$\begin{aligned} \Gamma^k_{mn}\boldsymbol{\gamma}_k = \partial_n \boldsymbol{\gamma}_m &= e_n{}^\mu \frac{\partial e_m{}^\kappa \boldsymbol{g}_\kappa}{\partial x^\nu} \quad \text{is } \textit{not} \text{ a tensor} \\ &= e_n{}^\mu \frac{\partial e_m{}^\kappa}{\partial x^\nu} \boldsymbol{g}_\kappa + e_n{}^\mu \, e_m{}^\kappa \frac{\partial \boldsymbol{g}_\kappa}{\partial x^\nu} \\ &= d_{kmn} \, e^{k\kappa} \, \boldsymbol{g}_\kappa + e_n{}^\mu \, e_m{}^\kappa \, \Gamma^\lambda_{\kappa\nu} \, \boldsymbol{g}_\lambda \ . \end{aligned} \tag{31}$$

Thus the relation is

$$\boxed{\Gamma_{lmn} - d_{lmn} = e_l{}^\lambda \, e_m{}^\mu \, e_n{}^\nu \, \Gamma_{\lambda\mu\nu}} \quad \text{is } \textit{not} \text{ a tensor} \tag{32}$$

where

$$\Gamma_{lmn} \equiv \gamma_{lk} \, \Gamma^k_{mn} \ . \tag{33}$$

5.13 Torsion tensor

The **torsion tensor** S^m_{kl}, which GR assumes to vanish, is defined in the usual way by the commutator of the covariant derivative acting on a scalar Φ

$$\boxed{[D_k, D_l] \, \Phi = S^m_{kl} \, \partial_m \Phi} \quad \textit{is} \text{ a tensor} \ . \tag{34}$$

The expression (29) for the covariant derivatives coupled with the commutator (20) of directed derivatives shows that the torsion tensor is

$$\boxed{S^m_{kl} = \Gamma^m_{kl} - \Gamma^m_{lk} - d^m_{kl} + d^m_{lk}} \quad \textit{is} \text{ a tensor} \tag{35}$$

where d^m_{kl} are the vierbein derivatives defined by equation (21). The torsion tensor S^m_{kl} is antisymmetric in $k \leftrightarrow l$, as is evident from its definition (34).

5.14 No-torsion condition

GR assumes vanishing torsion. Then equation (35) implies the no-torsion condition

$$\boxed{\Gamma_{mkl} - d_{mkl} = \Gamma_{mlk} - d_{mlk}} \quad \text{is } \textit{not} \text{ a tensor} \ . \tag{36}$$

In view of the relation (32) between tetrad and coordinate connections, the no-torsion condition (36) is equivalent to the usual symmetry condition $\Gamma_{\mu\kappa\lambda} = \Gamma_{\mu\lambda\kappa}$ on the coordinate frame connections, as it should be.

5.15 Antisymmetry of the connection coefficients

The directed derivative of the tetrad metric is

$$\begin{aligned}
\partial_n \gamma_{lm} &= \partial_n (\boldsymbol{\gamma}_l \cdot \boldsymbol{\gamma}_m) \\
&= \boldsymbol{\gamma}_l \cdot \partial_n \boldsymbol{\gamma}_m + \boldsymbol{\gamma}_m \cdot \partial_n \boldsymbol{\gamma}_l \\
&= \Gamma_{lmn} + \Gamma_{mln} \ .
\end{aligned} \tag{37}$$

In the great majority of cases, the tetrad metric is chosen to be a constant. This is true for example if the tetrad is orthonormal, so that the tetrad metric is the Minkowski metric. If the tetrad metric is constant, then all derivatives of the tetrad metric vanish, and then equation (37) shows that the tetrad connections are antisymmetric in their first two indices

$$\boxed{\Gamma_{lmn} = -\Gamma_{mln}} \ . \tag{38}$$

This antisymmetry reflects the fact that Γ_{lmn} is the generator of a Lorentz transformation for each n.

5.16 Connection coefficients in terms of the vierbein

In the general case of non-constant tetrad metric, and non-vanishing torsion, the following manipulation

$$\begin{aligned}
\partial_n \gamma_{lm} + \partial_m \gamma_{ln} - \partial_l \gamma_{mn} &= \Gamma_{lmn} + \Gamma_{mln} + \Gamma_{lnm} + \Gamma_{nlm} - \Gamma_{mnl} - \Gamma_{nml} \qquad (39) \\
&= 2\,\Gamma_{lmn} - S_{lmn} - S_{mnl} - S_{nml} - d_{lmn} + d_{lnm} - d_{mnl} + d_{mln} - d_{nml} + d_{nlm}
\end{aligned}$$

implies that the tetrad connections Γ_{lmn} are given in terms of the derivatives $\partial_n \gamma_{lm}$ of the tetrad metric, the torsion S_{lmn}, and the vierbein derivatives d_{lmn} by

$$\begin{aligned}
\Gamma_{lmn} = \ & \frac{1}{2} \big(\partial_n \gamma_{lm} + \partial_m \gamma_{ln} - \partial_l \gamma_{mn} + S_{lmn} + S_{mnl} + S_{nml} \\
& + d_{lmn} - d_{lnm} + d_{mnl} - d_{mln} + d_{nml} - d_{nlm} \big) \quad \text{is } \textit{not} \text{ a tensor} \ .
\end{aligned} \tag{40}$$

If torsion vanishes, as GR assumes, and if furthermore the tetrad metric is constant, then equation (40) simplifies to the following expression for the tetrad connections in terms of the vierbein derivatives d_{lmn} defined by (21)

$$\boxed{\Gamma_{lmn} = \frac{1}{2} \left(d_{lmn} - d_{lnm} + d_{mnl} - d_{mln} + d_{nml} - d_{nlm} \right)} \quad \text{is } \textit{not} \text{ a tensor} \ . \tag{41}$$

This is the formula that allows connection coefficients to be calculated from the vierbein.

5.17 Riemann curvature tensor

The **Riemann curvature tensor** R_{klmn} is defined in the usual way by the commutator of the covariant derivative acting on a contravariant 4-vector

$$\boxed{[D_k, D_l]\, A_m = R_{klmn} A^n} \quad \textit{is} \text{ a tensor} \ . \tag{42}$$

THE DEPENDENCE ON TORSION IS WRONG. IT SHOULD AGREE WITH EQ (105) IN THE COORDINATE FORMALISM.

The expression (29) for the covariant derivative coupled with the torsion equation (34) yields the following formula for the Riemann tensor in terms of connection coefficients, for the general case of non-vanishing torsion:

$$R_{klmn} = \partial_k \Gamma_{mnl} - \partial_l \Gamma_{mnk} + \Gamma^a_{ml} \Gamma_{ank} - \Gamma^a_{mk} \Gamma_{anl} + (\Gamma^a_{kl} - \Gamma^a_{lk} - S^a_{kl}) \Gamma_{mna} \quad \textit{is a tensor} \ . \tag{43}$$

The formula has the extra terms $(\Gamma^a_{kl} - \Gamma^a_{lk} - S^a_{kl}) \Gamma_{mna}$ compared to the usual formula for the coordinate-frame Riemann tensor $R_{\kappa\lambda\mu\nu}$. If torsion vanishes, as GR assumes, then

$$\boxed{R_{klmn} = \partial_k \Gamma_{mnl} - \partial_l \Gamma_{mnk} + \Gamma^a_{ml} \Gamma_{ank} - \Gamma^a_{mk} \Gamma_{anl} + (\Gamma^a_{kl} - \Gamma^a_{lk}) \Gamma_{mna}} \quad \textit{is a tensor} \ . \tag{44}$$

The symmetries of the tetrad-frame Riemann tensor are the same as those of the coordinate-frame Riemann tensor. For vanishing torsion, these are

$$R_{([kl][mn])} \ , \tag{45}$$

$$R_{klmn} + R_{knlm} + R_{kmnl} = 0 \ . \tag{46}$$

5.18 Ricci, Einstein, Weyl, Bianchi

The usual suite of formulae leading to Einstein's equations apply. Since all the quantities are tensors, and all the equations are tensor equations, their form follows immediately from their coordinate counterparts.

Ricci tensor:

$$\boxed{R_{km} \equiv \gamma^{ln} R_{klmn}} \ . \tag{47}$$

Ricci scalar:

$$\boxed{R \equiv \gamma^{km} R_{km}} \ . \tag{48}$$

Einstein tensor:

$$\boxed{G_{km} \equiv R_{km} - \frac{1}{2} R \gamma_{km}} \ . \tag{49}$$

Einstein's equations:

$$\boxed{G_{km} = 8\pi G T_{km}} \ . \tag{50}$$

Weyl tensor:

$$\boxed{C_{klmn} \equiv R_{klmn} - \frac{1}{2} \left(\gamma_{km} R_{ln} - \gamma_{kn} R_{lm} + \gamma_{ln} R_{km} - \gamma_{lm} R_{kn} \right) + \frac{1}{6} \left(\gamma_{km} \gamma_{ln} - \gamma_{kn} \gamma_{lm} \right)} \ . \tag{51}$$

Bianchi identities:

$$\boxed{D_k R_{lmnp} + D_l R_{mknp} + D_m R_{klnp} = 0} \ , \tag{52}$$

which most importantly imply covariant conservation of the Einstein tensor, hence conservation of energy-momentum

$$\boxed{D^k T_{km} = 0} \ . \tag{53}$$

5.19 Electromagnetism

5.19.1 Electromagnetic field

The electromagnetic field is a bivector field (an antisymmetric tensor) F^{mn} whose 6 components comprise the electric field $\boldsymbol{E} = E_i$ and magnetic field $\boldsymbol{B} = B_i$. In an orthonormal tetrad,

$$F^{mn} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} . \tag{54}$$

5.19.2 Lorentz force law

In the presence of an electromagnetic field F^{mn}, the general relativistic equation of motion for the 4-velocity $u^m \equiv dx^m/d\tau$ of a particle of mass m and charge q is modified by the addition of a **Lorentz force** $qF^m{}_n u^n$

$$m\frac{Du^m}{D\tau} = qF^m{}_n u^n \ . \tag{55}$$

In the absence of gravitational fields, so $D/D\tau = d/d\tau$, and with $u^m = u^t\{1, \boldsymbol{v}\}$ where $\boldsymbol{v}$ is the 3-velocity, the spatial components of equation (55) reduce to [note that $d/dt = (1/u^t)d/d\tau$]

$$m\frac{du^i}{dt} = q\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right) \quad i = 1, 2, 3 \tag{56}$$

which is the classical special relativistic Lorentz force law. The signs in the expression (54) for F^{mn} in terms of $\boldsymbol{E} = E_i$ and $\boldsymbol{B} = B_i$ are arranged to agree with the classical law (56).

5.19.3 Maxwell's equations

The source-free Maxwell's equations are

$$\boxed{D^l F^{mn} + D^m F^{nl} + D^n F^{lm} = 0} \ , \tag{57}$$

while the soured Maxwell's equations are

$$\boxed{D_m F^{mn} = 4\pi j^n} \ , \tag{58}$$

where j^n is the electric 4-current. The sourced Maxwell's equations (58) coupled with the antisymmetry of the electromagnetic field tensor F^{mn} ensure conservation of electric charge

$$\boxed{D_n j^n = 0} \ . \tag{59}$$

5.19.4 Electromagnetic energy-momentum tensor

The energy-momentum tensor of an electromagnetic field F^{mn} is

$$\boxed{T_e^{mn} = \frac{1}{4\pi}\left(-F^m{}_k F^{nk} + \frac{1}{4}\gamma^{mn}F_{kl}F^{kl}\right)} \, . \tag{60}$$

5.20 Gullstrand-Painlevé river

The aim of this section is to show rigorously how the Gullstrand-Painlevé metric paints a picture of space falling like a river into a Schwarzschild or Reissner-Nordström black hole. The river has two key features: first, the river flows in Galilean fashion through a flat Galilean background; and second, as a freely-falling fishy swims through the river, its 4-velocity, or more generally any 4-vector attached to it, evolves by a series of infinitesimal Lorentz boosts induced by the change in the velocity of the river from place to place. Because the river moves in Galilean fashion, it can, and inside the horizon does, move faster than light through the background coordinates. However, objects moving in the river move according to the rules of special relativity, and so cannot move faster than light through the river.

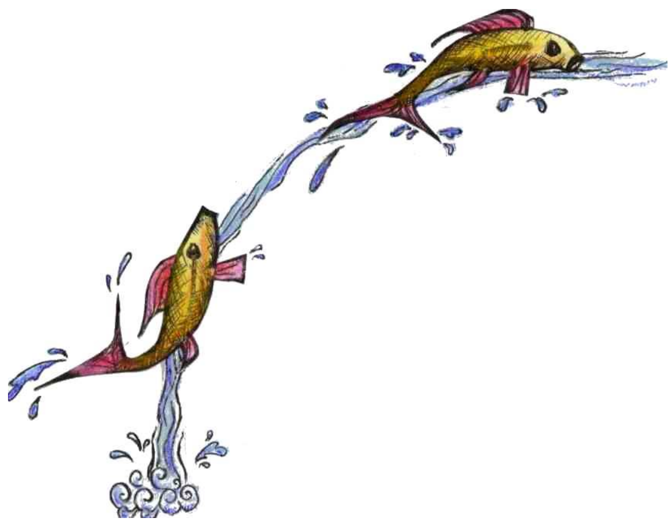

Figure 1: The fish upstream can make way against the current, but the fish downstream is swept to the bottom of the waterfall.

5.20.1 Gullstrand-Painlevé-Cartesian coordinates

In place of a polar coordinate system, introduce a Cartesian coordinate system $x^\mu \equiv \{t_{\rm ff}, x^i\} \equiv \{t_{\rm ff}, x, y, z\}$. The Gullstrand-Painlevé metric in these Cartesian coordinates is

$$\boxed{ds^2 = dt_{\rm ff}^2 - \delta_{ij}(dx^i - \beta^i dt_{\rm ff})(dx^j - \beta^j dt_{\rm ff})} \tag{61}$$

with implicit summation over spatial indices $i, j = x, y, z$. The β^i in the metric (61) are the components of the radial infall velocity expressed in Cartesian coordinates

$$\beta^i = \beta \left\{ \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\} \ . \tag{62}$$

Physically, t_{ff} is the proper time experienced by observers who free-fall radially from zero velocity at infinity, and β^i constitute the spatial components of their 4-velocity

$$\beta^i = \frac{dx^i}{dt_{\text{ff}}} \ . \tag{63}$$

For the Schwarzschild or Reissner-Nordström geometry, the infall velocity is

$$\beta = -\sqrt{\frac{2M(r)}{r}} \tag{64}$$

where $M(r)$ is the interior mass within radius r, which is the mass M at infinity minus the mass $Q^2/2r$ in the electric field outside r,

$$M(r) = M - \frac{Q^2}{2r} \ . \tag{65}$$

The Gullstrand-Painlevé metric (61) encodes an inverse vierbein $e^m{}_\mu$ through

$$ds^2 = \eta_{mn} \, e^m{}_\mu \, e^n{}_\nu \, dx^\mu \, dx^\nu \ . \tag{66}$$

The vierbein $e_m{}^\mu$ and inverse vierbein $e^m{}_\mu$ are explicitly

$$e_m{}^\mu = \begin{pmatrix} 1 & \beta^x & \beta^y & \beta^z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ , \quad e^m{}_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\beta^x & 1 & 0 & 0 \\ -\beta^y & 0 & 1 & 0 \\ -\beta^z & 0 & 0 & 1 \end{pmatrix} \ . \tag{67}$$

5.20.2 Gullstrand-Painlevé-Cartesian tetrad

The tetrad and coordinate axes of the Gullstrand-Painlevé tetrad are related to each other by

$$\boldsymbol{\gamma}_m = e_m{}^\mu \, \boldsymbol{g}_\mu \ , \quad \boldsymbol{g}_\mu = e^m{}_\mu \, \boldsymbol{\gamma}_m \ . \tag{68}$$

Explicitly, the tetrad axes $\boldsymbol{\gamma}_m$ are related to the coordinate tangent axes $\boldsymbol{g}_\mu$ by

$$\boxed{\boldsymbol{\gamma}_{t_{\text{ff}}} = \boldsymbol{g}_{t_{\text{ff}}} + \beta^i \boldsymbol{g}_i \ , \quad \boldsymbol{\gamma}_i = \boldsymbol{g}_i} \ . \tag{69}$$

Physically, the Gullstrand-Painlevé tetrad (69) are the axes of locally inertial orthonormal frames that coincide with the axes of the Cartesian rest frame at infinity, and are attached to observers who free-fall radially, without rotating, starting from zero velocity and zero angular

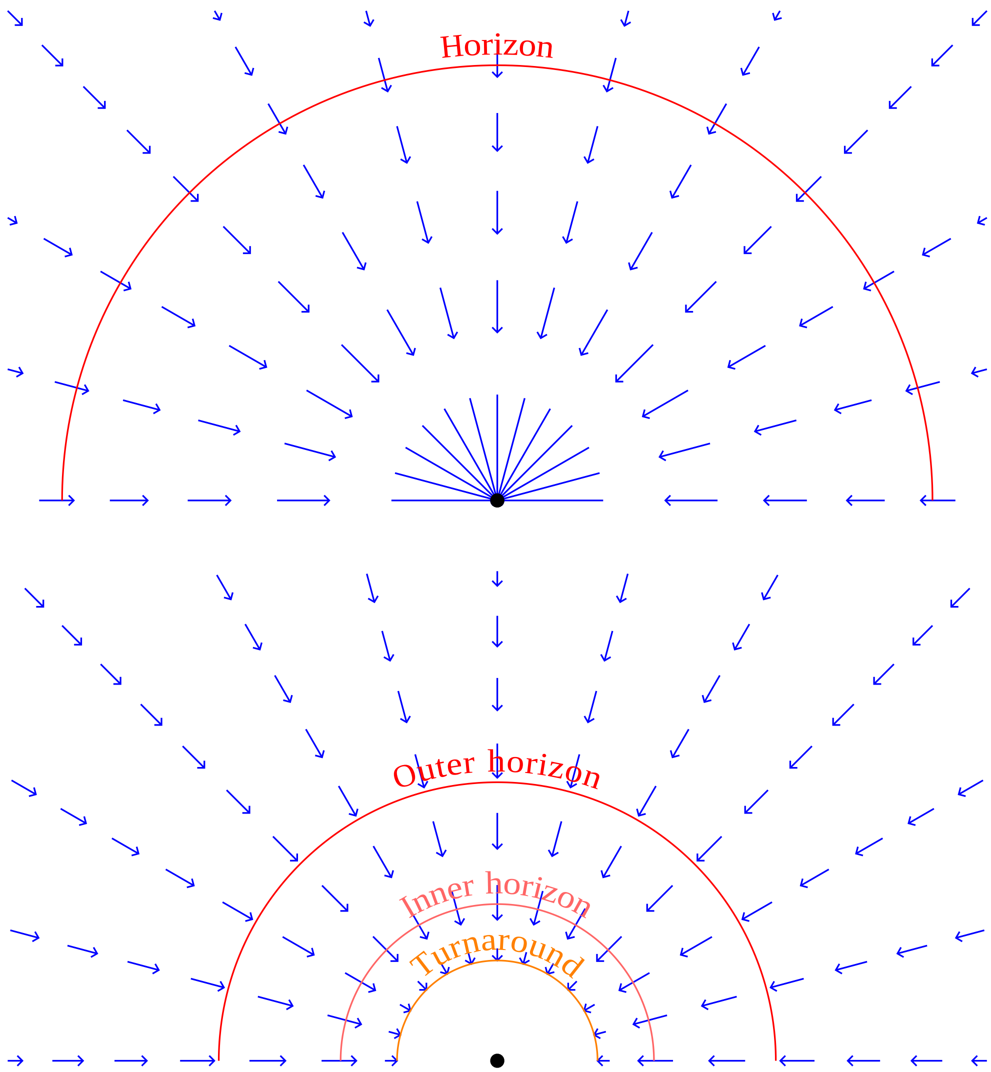

Figure 2: Velocity fields in (upper panel) a Schwarzschild black hole, and (lower panel) a Reissner-Nordström black hole with electric charge $Q = 0.96$.

momentum at infinity. The fact that the tetrad axes $\boldsymbol{\gamma}_m$ are parallel-transported, without precessing, along the worldlines of the radially free-falling observers can be confirmed by checking that

$$\frac{d\boldsymbol{\gamma}_m}{dt_{\rm ff}} = u^\nu \frac{\partial \boldsymbol{\gamma}_m}{\partial x^\nu} = 0 \tag{70}$$

where $u^\nu \equiv dx^\nu / dt_{\rm ff} = \{1, \beta^i\}$ is the coordinate 4-velocity of the radially free-falling observers.

Remarkably, the transformation (69) from coordinate to tetrad axes is just a Galilean transformation of space and time, which shifts the time axis by velocity β along the direction of motion, but which leaves unchanged both the time component of the time axis and all the spatial axes. In other words, the black hole behaves as if it were a river of space that flows radially inward through Galilean space and time at the Newtonian escape velocity.

5.20.3 Gullstrand-Painlevé fishies

The non-zero tetrad connection coefficients corresponding to the Gullstrand-Painlevé vierbein (67) prove to be given by the gradient of the infall velocity

$$\Gamma^{t_{\rm ff}}_{ij} = \frac{\partial \beta^i}{\partial x^j} \quad (i, j = x, y, z) \ . \tag{71}$$

Consider a fishy swimming in the Gullstrand-Painlevé river, with some arbitrary 4-velocity u^m, and consider a 4-vector p^k attached to the fishy. If the fishy is following a geodesic, then the equation of motion for p^k is

$$\frac{dp^k}{d\tau} + \Gamma^k_{mn} u^n p^m = 0 \ . \tag{72}$$

With the connections (71), the equation of motion (72) translates to (the following equations assume implicit summation over repeated spatial indices, even though the indices are not always one up one down)

$$\frac{dp^{t_{\rm ff}}}{d\tau} = - \frac{\partial \beta^i}{\partial x^j} u^j p^i \ , \quad \frac{dp^i}{d\tau} = - \frac{\partial \beta^i}{\partial x^j} u^j p^{t_{\rm ff}} \ . \tag{73}$$

In a small time $\delta\tau$, the fishy moves a proper distance $\delta\xi^m \equiv u^m \delta\tau$ relative to the infalling river. This proper distance $\delta\xi^m = e^m{}_\mu \delta x^\mu = \delta^m_\mu (\delta x^\mu - \beta^\mu \delta t_{\rm ff}) = \delta x^m - \beta^m \delta\tau$ equals the distance δx^m moved relative to the background Gullstrand-Painlevé-Cartesian coordinates, minus the distance $\beta^m \delta\tau$ moved by the river. From the fishy's perspective, the velocity of the river changes during this motion by an amount

$$\delta\beta^i = \delta\xi^j \, \frac{\partial \beta^i}{\partial x^j} \tag{74}$$

in which the sum over j can be taken over spatial indices only because, thanks to time translation symmetry, the velocity β^i has no explicit dependence on time $t_{\rm ff}$. According to the equation of motion (73), the 4-vector p^k changes by

$$p^{t_{\rm ff}} \to p^{t_{\rm ff}} - \delta\beta^i \, p^i \ , \quad p^i \to p^i - \delta\beta^i \, p^{t_{\rm ff}} \ . \tag{75}$$

But this is nothing more than an infinitesimal Lorentz boost by a velocity change $\delta\beta^i$. This shows that a fishy swimming in the river follows the rules of special relativity, being Lorentz boosted by tidal changes $\delta\beta^i$ in the river velocity from place to place.

Is it correct to interpret equation (74) as giving the change $\delta\beta^i$ in the river velocity seen by a fishy? Shouldn't the change in the river velocity really be

$$\delta\beta^i \stackrel{?}{=} \delta x^\nu \frac{\partial\beta^i}{\partial x^\nu} \tag{76}$$

where δx^ν is the full change in the coordinate position of the fishy? The answer is no. Part of the change (76) in the river velocity can be attributed to the change in the velocity of the river itself over the time $\delta\tau$, which is $\delta x^\nu_{\text{river}}\partial\beta^i/\partial x^\nu$ with $\delta x^\nu_{\text{river}} = \beta^\nu\delta\tau = \beta^\nu\delta t_{\text{ff}}$. The change in the velocity relative to the flowing river is

$$\delta\beta^i = (\delta x^\nu - \delta x^\nu_{\text{river}})\frac{\partial\beta^i}{\partial x^\nu} = (\delta x^\nu - \beta^\nu\delta t_{\text{ff}})\frac{\partial\beta^i}{\partial x^\nu} \tag{77}$$

which reproduces the earlier expression (74). Indeed, in the picture of fishies being carried by the river, it is essential to subtract the change in velocity of the river itself, as in equation (77), because otherwise fishies at rest in the river (going with the flow) would not continue to remain at rest in the river.

5.21 Doran river

The picture of space falling into a black hole like a river works also for rotating black holes. For Kerr-Newman rotating black holes, the counterpart of the Gullstrand-Painlevé metric is the Doran (2000) metric.

The river that falls into a rotating black hole has a mind-bending twist. One might have expected that the rotation of the black hole would be reflected by an infall velocity that spirals inward, but this is not the case. Instead, the river is characterized not merely by a velocity but also by a twist. The velocity and the twist together comprise a 6-dimensional river bivector ω_{km}, equation (89) below, whose electric part is the velocity, and whose magnetic part is the twist. Recall that the 6-dimensional group of Lorentz transformations is generated by a combination of 3-dimensional Lorentz boosts and 3-dimensional spatial rotations. A fishy that swims through the river is Lorentz boosted by tidal changes in the velocity, and rotated by tidal changes in the twist, equation (98).

Thanks to the twist, unlike the Gullstrand-Painlevé metric, the Doran metric is not spatially flat at constant free-fall time t_{ff}. Rather, the spatial metric is sheared in the azimuthal direction. Just as the velocity produces a Lorentz boost that makes the metric non-flat with respect to the time components, so also the twist produces a rotation that makes the metric non-flat with respect to the spatial components.

5.21.1 Doran-Cartesian coordinates

In place of the polar coordinates $\{r, \theta, \phi_{\text{ff}}\}$ of the Doran metric, introduce corresponding Doran-Cartesian coordinates $\{x, y, z\}$ with z taken along the rotation axis of the black hole

(the black hole rotates right-handedly about z, for positive spin parameter a)

$$x \equiv R \sin\theta \cos\phi_{\text{ff}} \ , \quad y \equiv R \sin\theta \sin\phi_{\text{ff}} \ , \quad z \equiv r \cos\theta \ . \tag{78}$$

The metric in Doran-Cartesian coordinates $x^\mu \equiv \{t_{\text{ff}}, x^i\} \equiv \{t_{\text{ff}}, x, y, z\}$, is

$$\boxed{ds^2 = dt_{\text{ff}}^2 - \delta_{ij} \left(dx^i - \beta^i \alpha_\kappa dx^\kappa\right) \left(dx^j - \beta^j \alpha_\lambda dx^\lambda\right)} \tag{79}$$

where α_μ is the rotational velocity vector

$$\alpha_\mu = \left\{1, \frac{ay}{R^2}, -\frac{ax}{R^2}, 0\right\} \ , \tag{80}$$

and β^μ is the infall velocity vector

$$\beta^\mu = \frac{\beta R}{\rho} \left\{0, \frac{xr}{R\rho}, \frac{yr}{R\rho}, \frac{zR}{r\rho}\right\} \ . \tag{81}$$

The rotational velocity and infall velocity vectors are orthogonal

$$\alpha_\mu \beta^\mu = 0 \ . \tag{82}$$

For the Kerr-Newman metric, the infall velocity β is

$$\beta = \mp \frac{\sqrt{2Mr - Q^2}}{R} \tag{83}$$

with $-$ for black hole (infalling), $+$ for white hole (outfalling) solutions. Horizons occur where $|\beta| = 1$, with $\beta = -1$ for black hole horizons, $\beta = 1$ for white hole horizons.

The Doran-Cartesian metric (79) encodes a vierbein $e_m{}^\mu$ and inverse vierbein $e^m{}_\mu$

$$e_m{}^\mu = \delta_m^\mu + \alpha_m \beta^\mu \ , \quad e^m{}_\mu = \delta_\mu^m - \alpha_\mu \beta^m \ . \tag{84}$$

Here the tetrad-frame components α_m of the rotational velocity vector and β^m of the infall velocity vector are

$$\alpha_m = e_m{}^\mu \alpha_\mu = \delta_m^\mu \alpha_\mu \ , \quad \beta^m = e^m{}_\mu \beta^\mu = \delta_\mu^m \beta^\mu \ , \tag{85}$$

which works thanks to the orthogonality (82) of α_μ and β^μ. Equation (85) says that the covariant tetrad-frame components of the rotational velocity vector α are the same as its covariant coordinate-frame components in the Doran-Cartesian coordinate system, $\alpha_m = \alpha_\mu$, and likewise the contravariant tetrad-frame components of the infall velocity vector β are the same as its contravariant coordinate-frame components, $\beta^m = \beta^\mu$.

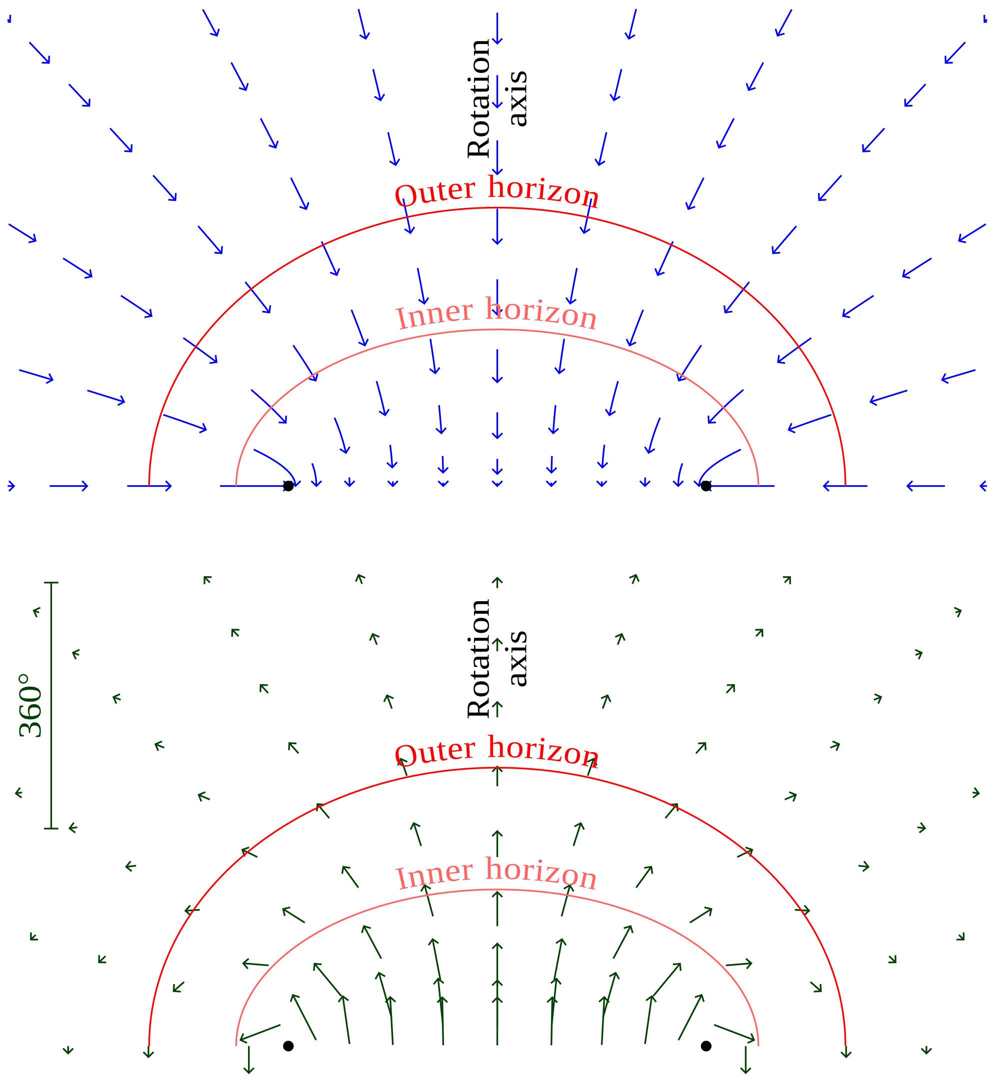

Figure 3: (Upper panel) velocity β^i and (lower panel) twist μ^i vector fields for a Kerr black hole with spin parameter $a = 0.96$. Both vectors lie, as shown, in the plane of constant free-fall azimuthal angle $\phi_{\rm ff}$.

5.21.2 Doran-Cartesian tetrad

Like the Gullstrand-Painlevé tetrad, the Doran-Cartesian tetrad $\boldsymbol{\gamma}_m \equiv \{\boldsymbol{\gamma}_{t_{\rm ff}}, \boldsymbol{\gamma}_x, \boldsymbol{\gamma}_y, \boldsymbol{\gamma}_z\}$ is aligned with the Cartesian rest frame at infinity, and is parallel-transported, without precessing, by observers who free-fall from zero velocity and zero angular momentum at infinity. Let $\|$ and $\perp$ subscripts denote horizontal radial and azimuthal directions respectively, so that

$$\begin{aligned} \boldsymbol{\gamma}_\| &\equiv \cos\phi_{\rm ff}\,\boldsymbol{\gamma}_x + \sin\phi_{\rm ff}\,\boldsymbol{\gamma}_y \ , \quad \boldsymbol{\gamma}_\perp \equiv - \sin\phi_{\rm ff}\,\boldsymbol{\gamma}_x + \cos\phi_{\rm ff}\,\boldsymbol{\gamma}_y \ , \\ \boldsymbol{g}_\| &\equiv \cos\phi_{\rm ff}\,\boldsymbol{g}_x + \sin\phi_{\rm ff}\,\boldsymbol{g}_y \ , \quad \boldsymbol{g}_\perp \equiv - \sin\phi_{\rm ff}\,\boldsymbol{g}_x + \cos\phi_{\rm ff}\,\boldsymbol{g}_y \ . \end{aligned} \tag{86}$$

Then the relation between Doran-Cartesian tetrad axes $\boldsymbol{\gamma}_m$ and the tangent axes $\boldsymbol{g}_\mu$ of the Doran-Cartesian metric (79) is

$$\boldsymbol{\gamma}_{t_{\rm ff}} = \boldsymbol{g}_{t_{\rm ff}} + \beta^i \boldsymbol{g}_i \ , \tag{87a}$$

$$\boldsymbol{\gamma}_\| = \boldsymbol{g}_\| \ , \tag{87b}$$

$$\boldsymbol{\gamma}_\perp = \boldsymbol{g}_\perp - \frac{a\sin\theta}{R}\beta^i \boldsymbol{g}_i \ , \tag{87c}$$

$$\boldsymbol{\gamma}_z = \boldsymbol{g}_z \ . \tag{87d}$$

The relations (87) resemble those (69) of the Gullstrand-Painlevé tetrad, except that the azimuthal tetrad axis $\boldsymbol{\gamma}_\perp$ is shifted radially relative to the azimuthal tangent axis $\boldsymbol{g}_\perp$. This shift reflects the fact that, unlike the Gullstrand-Painlevé metric, the Doran metric is not spatially flat at constant free-fall time.

5.21.3 Doran fishies

The tetrad-frame connections equal the ordinary partial derivatives in Doran-Cartesian coordinates of a bivector (antisymmetric tensor) ω_{km}

$$\boxed{\Gamma_{kmn} = - \frac{\partial \omega_{km}}{\partial x^n}} \tag{88}$$

which I call the river field because it encapsulates all the properties of the infalling river of space. The bivector river field ω_{km} is

$$\boxed{\omega_{km} = \alpha_k \beta_m - \alpha_m \beta_k + \varepsilon_{t_{\rm ff} kmi}\,\zeta^i} \tag{89}$$

where $\beta_m = \eta_{mn}\beta^m$, the totally antisymmetric tensor ε_{klmn} is normalized so that $\varepsilon_{t_{\rm ff}xyz} = -1$, and the vector ζ^i points vertically upward along the rotation axis of the black hole

$$\zeta^i \equiv \{0, 0, 0, \zeta\} \ , \quad \zeta \equiv a \int_\infty^r \frac{\beta\, dr}{R^2} \ . \tag{90}$$

The electric part of ω_{km}, where one of the indices is time $t_{\rm ff}$, constitutes the velocity vector β^i

$$\omega_{t_{\rm ff} i} = \beta^i \tag{91}$$

while the magnetic part of ω_{km}, where both indices are spatial, constitutes the twist vector μ^i defined by

$$\mu^i \equiv \tfrac{1}{2}\, \varepsilon^{t_{\rm ff} ikm} \omega_{km} = \varepsilon^{t_{\rm ff} ikm} \alpha_k \beta_m + \zeta^i \ . \tag{92}$$

The sense of the twist is that induces a right-handed rotation about an axis equal to the direction of μ^i by an angle equal to the magnitude of μ^i. In 3-vector notation, with $\boldsymbol{\mu} \equiv \mu^i$, $\boldsymbol{\alpha} \equiv \alpha_i$, $\boldsymbol{\beta} \equiv \beta^i$, $\boldsymbol{\zeta} \equiv \zeta^i$,

$$\boldsymbol{\mu} \equiv \boldsymbol{\alpha} \times \boldsymbol{\beta} + \boldsymbol{\zeta} \ . \tag{93}$$

In terms of the velocity and twist vectors, the river field ω_{km} is

$$\omega_{km} = \begin{pmatrix} 0 & -\beta^x & -\beta^y & -\beta^z \\ \beta^x & 0 & -\mu^z & \mu^y \\ \beta^y & \mu^z & 0 & -\mu^x \\ \beta^z & -\mu^y & \mu^x & 0 \end{pmatrix} \ . \tag{94}$$

Note that the sign of the electric part $\boldsymbol{\beta}$ of ω_{km} is opposite to the sign of the analogous electric field $\boldsymbol{E}$ associated with an electromagnetic field F_{km}; but the adopted signs are natural in that the river field induces boosts in the direction of the velocity β^i, and right-handed rotations about the twist μ^i. Like a static electric field, the velocity vector β^i is the gradient of a potential

$$\beta^i = \frac{\partial}{\partial x^i} \int^r \beta \, dr \ , \tag{95}$$

but unlike a magnetic field the twist vector μ^i is not pure curl: rather, it is $\mu^i + \zeta^i$ that is pure curl.

With the tetrad connection coefficients given by equation (88), the equation of motion (72) for a 4-vector p^k attached to a fishy following a geodesic in the Doran river translates to

$$\frac{dp^k}{d\tau} = \frac{\partial \omega^k{}_m}{\partial x^n} u^n p^m \ . \tag{96}$$

In a proper time $\delta\tau$, the fishy moves a proper distance $\delta\xi^m \equiv u^m \delta\tau$ relative to the background Doran-Cartesian coordinates. As a result, the fishy sees a tidal change $\delta\omega^k{}_m$ in the river field

$$\delta\omega^k{}_m = \delta\xi^n \frac{\partial \omega^k{}_m}{\partial x^n} \ . \tag{97}$$

Consequently the 4-vector p^k is changed by

$$p^k \rightarrow p^k + \delta\omega^k{}_m \, p^m \ . \tag{98}$$

But equation (98) corresponds to a Lorentz boost by $\delta\beta^i$ and a rotation by $\delta\mu^i$.

As discussed previously with regard to the Gullstrand-Painlevé river, §5.20.3, the tidal change $\delta\omega^k{}_m$, equation (97), in the river field seen by a fishy is not the full change $\delta x^\nu \, \partial\omega^k{}_m / \partial x^\nu$ relative to the background coordinates, but rather the change relative to the river

$$\delta\omega^k{}_m = (\delta x^\nu - \delta x^\nu_{\rm river}) \frac{\partial \omega^k{}_m}{\partial x^\nu} = \left[\delta x^\nu - \beta^\nu (\delta t_{\rm ff} - a \sin^2\!\theta \, \delta\phi_{\rm ff})\right] \frac{\partial \omega^k{}_m}{\partial x^\nu} \tag{99}$$

with the change in the velocity and twist of the river itself subtracted off.

5.22 Boyer-Lindquist tetrad

The Kerr-Newman metric has a special orthonormal tetrad, aligned with the (ingoing or outgoing) principal null congruences, with respect to which the electromagnetic, energy-momentum, and Weyl tensors take particularly simple forms. The tetrad is the Boyer-Linquist orthonormal tetrad, encoded in the Boyer-Lindquist metric

$$\boxed{ds^2 = \frac{\Delta}{\rho^2}\left(dt - a\sin^2\!\theta\, d\phi\right)^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - \frac{R^4\sin^2\!\theta}{\rho^2}\left(d\phi - \frac{a}{R^2}dt\right)^2} \tag{100}$$

where

$$R \equiv \sqrt{r^2 + a^2}\ , \quad \rho \equiv \sqrt{r^2 + a^2\cos^2\!\theta}\ , \quad \Delta \equiv R^2 - 2Mr + Q^2 = R^2(1-\beta^2)\ . \tag{101}$$

Explicitly, the vierbein $e_m{}^\mu$ of the Boyer-Linquist orthonormal tetrad is

$$e_m{}^\mu = \begin{pmatrix} R/\left[\rho(1-\beta^2)^{1/2}\right] & 0 & 0 & a/\left[R\rho(1-\beta^2)^{1/2}\right] \\ 0 & R(1-\beta^2)^{1/2}/\rho & 0 & 0 \\ 0 & 0 & 1/\rho & 0 \\ a\,\sin\theta/\rho & 0 & 0 & 1/\left(\rho\,\sin\theta\right) \end{pmatrix}\ , \tag{102}$$

with inverse vierbein $e^m{}_\mu$

$$e^m{}_\mu = \begin{pmatrix} R(1-\beta^2)^{1/2}/\rho & 0 & 0 & -\,aR\sin^2\!\theta(1-\beta^2)^{1/2}/\rho \\ 0 & \rho/\left[R(1-\beta^2)^{1/2}\right] & 0 & 0 \\ 0 & 0 & \rho & 0 \\ -\,a\sin\theta/\rho & 0 & 0 & R^2\sin\theta/\rho \end{pmatrix}\ . \tag{103}$$

With respect to this tetrad, only the radial electric field E_r and magnetic field B_r are non-vanishing, and they are given by the complex combination

$$E_r + i\,B_r = \frac{Q}{(r - ia\cos\theta)^2}\ , \tag{104}$$

or explicitly

$$E_r = \frac{Q\,(r^2 - a^2\cos^2\!\theta)}{\rho^4}\ , \quad B_r = \frac{2Qar\cos\theta}{\rho^4}\ . \tag{105}$$

The electrogmagnetic field (104) satisfies Maxwell's equations (57) and (58) with zero electric current, $j^n = 0$.

The non-vanishing components of the tetrad-frame Einstein tensor G_{mn} are

$$G_{mn} = \frac{Q^2}{\rho^4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\ . \tag{106}$$

The non-vanishing components of the tetrad-frame Weyl tensor C_{klmn} are

$$-\tfrac{1}{2}\, C_{trtr} = \tfrac{1}{2}\, C_{\theta\phi\theta\phi} = C_{t\theta t\theta} = C_{t\phi t\phi} = -\, C_{r\theta r\theta} = -\, C_{r\phi r\phi} = \mathrm{Re}\, C \ , \tag{107a}$$

$$\tfrac{1}{2}\, C_{tr\theta\phi} = C_{t\theta r\phi} = -\, C_{t\phi r\theta} = \mathrm{Im}\, C \ , \tag{107b}$$

where C is the complex Weyl scalar

$$C = -\frac{1}{(r - ia\cos\theta)^3} \left(M - \frac{Q^2}{r + ia\cos\theta} \right) \ . \tag{108}$$

5.23 General spherically symmetric spacetime

Even in so simple a case as a general spherically symmetric spacetime, it is not an easy matter to find a physically illuminating form of the Einstein equations. The following is the best that I know of.

5.23.1 Tetrad and vierbein

Choose the tetrad $\boldsymbol{\gamma}_m$ to be orthonormal, meaning that the scalar products of the tetrad axes constitute the Minkowski metric, $\boldsymbol{\gamma}_m \cdot \boldsymbol{\gamma}_n = \eta_{mn}$. Choose polar coordinates $x^\mu \equiv \{t, r, \theta, \phi\}$. Let r be the circumferential radius, so that the angular part of the metric is $r^2 do^2$, which is a gauge-invariant definition of r. Choose the transverse tetrad axes $\boldsymbol{\gamma}_\theta$ and $\boldsymbol{\gamma}_\phi$ to be aligned with the transverse coordinate axes $\boldsymbol{g}_\theta$ and $\boldsymbol{g}_\phi$. Orthonormality requires

$$\boldsymbol{\gamma}_\theta = \frac{1}{r}\,\boldsymbol{g}_\theta \ , \quad \boldsymbol{\gamma}_\phi = \frac{1}{r\sin\theta}\,\boldsymbol{g}_\phi \ . \tag{109}$$

So far all the choices have been standard and natural. Now for some less standard choices. Choose the radial tetrad axis $\boldsymbol{\gamma}_r$ to be aligned with the radial coordinate axis $\boldsymbol{g}_r$

$$\boldsymbol{\gamma}_r = \beta_r\,\boldsymbol{g}_r \tag{110}$$

where $\beta_r(t, r)$ is some arbitrary function of coordinate time t and radius r (the reason for the subscript r on β_r will become apparent momentarily). More generally, the radial tetrad axis $\boldsymbol{\gamma}_r$ could be taken to be some combination of the time and radial coordinate axes $\boldsymbol{g}_t$ and $\boldsymbol{g}_r$, but the choice (110) can always be effected by a suitable radial Lorentz boost. These choices (109)–(110) exhaust the Lorentz freedoms in the choice of tetrad. The tetrad time axis $\boldsymbol{\gamma}_t$ must be some combination of the time and radial coordinate axes $\boldsymbol{g}_t$ and $\boldsymbol{g}_r$

$$\boldsymbol{\gamma}_t = \frac{1}{\alpha}\,\boldsymbol{g}_t + \beta_t\,\boldsymbol{g}_r \tag{111}$$

where $\alpha(t, r)$ and $\beta_t(t, r)$ are some arbitrary functions of coordinate time t and radius r. Equations (109)–(111) imply that the vierbein ${e_m}^\mu$ and its inverse ${e^m}_\mu$ have been chosen to be

$$\boxed{{e_m}^\mu = \begin{pmatrix} 1/\alpha & \beta_t & 0 & 0 \\ 0 & \beta_r & 0 & 0 \\ 0 & 0 & 1/r & 0 \\ 0 & 0 & 0 & 1/(r\sin\theta) \end{pmatrix} \ , \quad {e^m}_\mu = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ -\,\alpha\,\beta_t/\beta_r & 1/\beta_r & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix}} \ . \tag{112}$$

The directed derivatives ∂_t and ∂_r along the time and radial tetrad axes $\boldsymbol{\gamma}_t$ and $\boldsymbol{\gamma}_r$ are

$$\partial_t = {e_t}^\mu \frac{\partial}{\partial x^\mu} = \frac{1}{\alpha}\frac{\partial}{\partial t} + \beta_t \frac{\partial}{\partial r} \ , \quad \partial_r = {e_r}^\mu \frac{\partial}{\partial x^\mu} = \beta_r \frac{\partial}{\partial r} \ . \tag{113}$$

The tetrad-frame 4-velocity u^m of a person at rest in the tetrad frame is by definition $u^m = \{1, 0, 0, 0\}$. It follows that the coordinate 4-velocity u^μ of such a person is

$$u^\mu = {e_m}^\mu u^m = {e_t}^\mu = \{1/\alpha, \beta_t, 0, 0\} \ . \tag{114}$$

The directed time derivative ∂_t is just the proper time derivative along the worldline of a person continuously at rest in the tetrad frame (and who is therefore not in free-fall, but accelerating with the tetrad frame), which follows from

$$\frac{d}{d\tau} = \frac{dx^\mu}{d\tau}\frac{\partial}{\partial x^\mu} = u^\mu \frac{\partial}{\partial x^\mu} = u^m \partial_m = \partial_t \ . \tag{115}$$

By contrast, the proper time derivative measured by a person who is instantaneously at rest in the tetrad frame, but is in free-fall, is the covariant time derivative

$$\frac{D}{D\tau} = \frac{dx^\mu}{d\tau} D_\mu = u^\mu D_\mu = u^m D_m = D_t \ . \tag{116}$$

Since the coordinate radius r has been defined to be the circumferential radius, a gauge-invariant definition, it follows that the tetrad-frame gradient ∂_m of the coordinate radius r is a tetrad-frame 4-vector (a coordinate gauge-invariant object)

$$\partial_m r = e_m{}^\mu \frac{\partial r}{\partial x^\mu} = e_m{}^r = \beta_m = \{\beta_t, \beta_r, 0, 0\} \quad \text{is a tetrad 4-vector} \ . \tag{117}$$

This accounts for the notation β_t and β_r introduced above. Since β_m is a tetrad 4-vector, its scalar product with itself must be a scalar. This scalar defines the **interior mass** $M(t, r)$, also called the Misner-Sharp mass, by

$$\boxed{1 - \frac{2M}{r} \equiv -\,\beta_m \beta^m = -\,\beta_t^2 + \beta_r^2} \quad \text{is a coordinate and tetrad scalar} \ . \tag{118}$$

The interpretation of M as the interior mass will become evident below, §5.23.9.

5.23.2 Coordinate metric

The coordinate metric $ds^2 = \eta_{mn} e^m{}_\mu e^n{}_\nu dx^\mu dx^\nu$ corresponding to the vierbein (112) is

$$\boxed{ds^2 = \alpha^2 dt^2 - \frac{1}{\beta_r^2}\left(dr - \beta_t\, \alpha\, dt\right)^2 - r^2 do^2} \ . \tag{119}$$

A person instantaneously at rest in the tetrad frame satisfies $dr/dt = \beta_t\, \alpha$ according to equation (114), so it follows from the metric (119) that the proper time τ of a person at rest in the tetrad frame is related to the coordinate time t by

$$d\tau = \alpha\, dt \quad \text{in tetrad rest frame} \ . \tag{120}$$

The metric (119) is a bit unconventional in that it is not diagonal: g_{tr} does not vanish. However, there are two good reasons to consider a non-diagonal metric. First, as discussed in §5.23.12, Einstein's equations take a more insightful form when expressed in a non-diagonal frame where β_t does not vanish, such as in the center-of-mass frame. Second, if a horizon is present, as in the case of black holes, and if the radial coordinate is taken to be the circumferential radius r, then a diagonal metric will have a coordinate singularity at the horizon, which is not ideal.

5.23.3 Rest diagonal coordinate metric

Although this is not the choice adopted here, the metric (119) can always be brought to diagonal form by a coordinate transformation $t \rightarrow t_\times$ (subscripted $\times$ for diagonal) of the time coordinate. The t–r part of the metric is

$$g_{tt}\, dt^2 + 2\, g_{tr}\, dt\, dr + g_{rr}\, dr^2 = \frac{1}{g_{tt}} \left[(g_{tt}\, dt + g_{tr}\, dr)^2 + (g_{tt} g_{rr} - g_{tr}^2) dr^2 \right] \ . \tag{121}$$

This can be diagonalized by choosing the time coordinate $t_\times$ such that

$$f\, dt_\times = g_{tt}\, dt + g_{tr}\, dr \tag{122}$$

for some integrating factor $f(t, r)$. Equation (122) can be solved by choosing $t_\times$ to be constant along integral curves

$$\frac{dr}{dt} = - \frac{g_{tt}}{g_{tr}} \ . \tag{123}$$

The resulting diagonal metric is

$$ds^2 = \alpha_\times^2 dt_\times^2 - \frac{dr^2}{1 - 2M/r} - r^2 do^2 \ . \tag{124}$$

The metric (124) corresponds physically to the case where the tetrad frame is taken to be at rest in the spatial coordinates, $\beta_t = 0$, as can be seen by comparing it to the earlier metric (119). The metric coefficient g_{rr} in the metric (124) follows from the fact that $\beta_r^2 = 1 - 2M/r$ when $\beta_t = 0$, equation (118). The transformed time coordinate $t_\times$ is unspecified up to a transformation $t_\times \rightarrow f(t_\times)$. If the spacetime is asymptotically flat at infinity, then a natural way to fix the transformation is to choose $t_\times$ to be the proper time at rest at infinity.

5.23.4 Comoving diagonal coordinate metric

The metric (119) can also be brought to diagonal form by a coordinate transformation $r \rightarrow r_\times$, where, analogously to equation (122), $r_\times$ is chosen to satisfy

$$f\, dr_\times = g_{tr}\, dt + g_{rr}\, dr \tag{125}$$

for some integrating factor $f(t, r)$. The new coordinate $r_\times$ is constant along the worldline of an object at rest in the tetrad frame, so $r_\times$ can be regarded as a kind of Lagrangian coordinate. For example, $r_\times$ could be chosen equal to the circumferential radius r at some fixed instant of coordinate time t (say $t = 0$). The metric in this Lagrangian coordinate system takes the form

$$ds^2 = \alpha^2 dt^2 - \lambda^2 dr_\times^2 - r^2 do^2 \tag{126}$$

where the circumferential radius $r(t, r_\times)$ is considered to be an implicit function of t and the Lagrangian radial coordinate $r_\times$. However, this is not the path followed in these notes.

5.23.5 Tetrad connections

Now turn the handle to proceed towards the Einstein equations. The tetrad connections coefficients Γ_{kmn} are

$$
\begin{aligned}
\Gamma_{trt} &= g \ , && (127a) \\
\Gamma_{trr} &= h \ , && (127b) \\
\Gamma_{t\theta\theta} = \Gamma_{t\phi\phi} &= \frac{\beta_t}{r} \ , && (127c) \\
\Gamma_{r\theta\theta} = \Gamma_{r\phi\phi} &= \frac{\beta_r}{r} \ , && (127d) \\
\Gamma_{\theta\phi\phi} &= \frac{\cot\theta}{r} \ , && (127e)
\end{aligned}
$$

where g is the proper radial acceleration (minus the gravitational force) experienced by a person at rest in the tetrad frame

$$\boxed{g \equiv \partial_r \ln \alpha} \ , \tag{128}$$

and h is the "Hubble parameter" of the radial flow, as measured in the tetrad rest frame, defined by

$$\boxed{h \equiv \beta_t \frac{\partial \ln \alpha}{\partial r} + \frac{\partial \beta_t}{\partial r} - \partial_t \ln \beta_r} \ . \tag{129}$$

The interpretation of g as a proper acceleration and h as a radial Hubble parameter goes as follows. The tetrad-frame 4-velocity u^m of a person at rest in the tetrad frame is by definition $u^m = \{1, 0, 0, 0\}$. If the person at rest were in free fall, then the proper acceleration would be zero, but because this is a general spherical spacetime, the tetrad frame is not necessarily in free fall. The proper acceleration experienced by a person continuously at rest in the tetrad frame is the proper time derivative $Du^m/D\tau$ of the 4-velocity, which is

$$\frac{Du^m}{D\tau} = u^n D_n u^m = u^t D_t u^m = u^t \left(\partial_t u^m + \Gamma^m_{tt} u^t\right) = \Gamma^m_{tt} = \{0, \Gamma^r_{tt}, 0, 0\} = \{0, g, 0, 0\} \ . \tag{130}$$

Similarly, a person at rest in the tetrad frame will measure the 4-velocity of an adjacent person at rest in the tetrad frame a small proper radial distance $\delta\xi^r$ away to differ by $\delta\xi^r D_r u^m$. The Hubble parameter of the radial flow is thus the covariant radial derivative $D_r u^m$, which is

$$D_r u^m = \partial_r u^m + \Gamma^m_{tr} u^t = \Gamma^m_{tr} = \{0, \Gamma^r_{tr}, 0, 0\} = \{0, h, 0, 0\} \ . \tag{131}$$

Since h is a kind of radial Hubble parameter, it can be useful to define a corresponding radial scale factor λ by

$$h \equiv \partial_t \ln \lambda \ . \tag{132}$$

The scale factor λ is the same as the λ in the comoving coordinate metric of equation (126). This is true because h is a tetrad connection and therefore coordinate gauge-invariant, and the metric (126) is related to the metric (119) being considered by a coordinate transformation $r \to r_\times$.

5.23.6 Riemann and Weyl tensors

The non-vanishing components of the tetrad-frame Riemann tensor R_{klmn} are

$$
\begin{aligned}
R_{trtr} &= \partial_t h - \partial_r g + h^2 - g^2 \ , && (133a)\\
R_{t\theta t\theta} = R_{t\phi t\phi} &= \frac{1}{r}\left(\partial_t \beta_t - \beta_r g\right) \ , && (133b)\\
R_{r\theta r\theta} = R_{r\phi r\phi} &= \frac{1}{r}\left(\partial_r \beta_r - \beta_t h\right) \ , && (133c)\\
R_{t\theta r\theta} = R_{t\phi r\phi} = R_{r\theta t\theta} = R_{r\phi t\phi} &= \frac{1}{r}\left(\partial_t \beta_r - \beta_t g\right) \ = \ \frac{1}{r}\left(\partial_r \beta_t - \beta_r h\right) \ , && (133d)\\
R_{\theta\phi\theta\phi} &= -\frac{2M}{r^3} \ . && (133e)
\end{aligned}
$$

The non-vanishing components of the tetrad frame Weyl tensor C_{klmn} are

$$
-\tfrac{1}{2}\,C_{trtr} = \tfrac{1}{2}\,C_{\theta\phi\theta\phi} = C_{t\theta t\theta} = C_{t\phi t\phi} = -\,C_{r\theta r\theta} = -\,C_{r\phi r\phi} = C \tag{134}
$$

where C is the Weyl scalar

$$
C \equiv \frac{1}{6}\left(-\,R_{trtr} + R_{t\theta t\theta} - R_{r\theta r\theta} + R_{\theta\phi\theta\phi}\right) = \frac{1}{6}\left(G^{tt} - G^{rr} + G^{\theta\theta}\right) - \frac{M}{r^3} \ . \tag{135}
$$

5.23.7 Einstein equations

The non-vanishing components of the tetrad-frame Einstein tensor G^{km} are

$$
\begin{aligned}
G^{tr} &= 2\,R_{t\theta r\theta} \ , && (136a)\\
G^{tt} &= -\,2\,R_{r\theta r\theta} - R_{\theta\phi\theta\phi} \ , && (136b)\\
G^{rr} &= -\,2\,R_{t\theta t\theta} + R_{\theta\phi\theta\phi} \ , && (136c)\\
G^{\theta\theta} = G^{\phi\phi} &= -\,R_{trtr} - R_{t\theta t\theta} + R_{r\theta r\theta} \ , && (136d)
\end{aligned}
$$

whence

$$
\begin{aligned}
G^{tr} &= \frac{2}{r}\left(\partial_t \beta_r - \beta_t g\right) && (137a)\\
&= \frac{2}{r}\left(\partial_r \beta_t - \beta_r h\right) \ , && (137b)\\
G^{tt} &= \frac{2}{r}\left(-\,\partial_r \beta_r + \beta_t h + \frac{M}{r^2}\right) \ , && (137c)\\
G^{rr} &= \frac{2}{r}\left(-\,\partial_t \beta_t + \beta_r g - \frac{M}{r^2}\right) \ , && (137d)\\
G^{\theta\theta} = G^{\phi\phi} &= \frac{1}{r}\,\partial_r\left(rg + \beta_r\right) - \frac{1}{r}\,\partial_t\left(rh + \beta_t\right) + g^2 - h^2 \ . && (137e)
\end{aligned}
$$

The Einstein equations in the tetrad frame

$$
G^{km} = 8\pi T^{km} \tag{138}
$$

imply that

$$
\begin{pmatrix} G^{tt} & G^{tr} & 0 & 0 \\ G^{tr} & G^{rr} & 0 & 0 \\ 0 & 0 & G^{\theta\theta} & 0 \\ 0 & 0 & 0 & G^{\phi\phi} \end{pmatrix} = 8\pi T^{mn} = 8\pi \begin{pmatrix} \rho & f & 0 & 0 \\ f & p & 0 & 0 \\ 0 & 0 & p_\perp & 0 \\ 0 & 0 & 0 & p_\perp \end{pmatrix} \tag{139}
$$

where $\rho \equiv T^{tt}$ is the proper energy density, $f \equiv T^{tr}$ is the proper radial energy flux, $p \equiv T^{rr}$ is the proper radial pressure, and $p_\perp \equiv T^{\theta\theta} = T^{\phi\phi}$ is the proper transverse pressure.

5.23.8 Choose your frame

So far the radial motion of the tetrad frame has been left unspecified. Any arbitrary choice can be made. For example, the tetrad frame could be chosen to be at rest,

$$
\beta_t = 0 \ , \tag{140}
$$

as in the Schwarzschild or Reissner-Nordström metrics. Alternatively, the tetrad frame could be chosen to be in free-fall,

$$
g = 0 \ , \tag{141}
$$

as in the Gullstrand-Painlevé metric. For situations where the spacetime contains matter, perhaps the most natural choice is the **center-of-mass frame**, defined to be the frame in which the energy flux f is zero

$$
G^{tr} = 8\pi f = 0 \ . \tag{142}
$$

Whatever the choice of radial tetrad frame, tetrad-frame quantities in different radial tetrad frames are related to each other by a radial Lorentz boost.

5.23.9 Interior mass

Equations (137c) with (137a), and (137d) with (137b), respectively, along with the definition (118) of the interior mass M, and the Einstein equations (139), imply

$$
p = \frac{1}{\beta_t}\left(-\frac{1}{4\pi r^2}\,\partial_t M - \beta_r f\right) \ , \tag{143a}
$$

$$
\rho = \frac{1}{\beta_r}\left(\frac{1}{4\pi r^2}\,\partial_r M - \beta_t f\right) \ . \tag{143b}
$$

In the center-of-mass frame, $f = 0$, these equations reduce to

$$
\partial_t M = -4\pi r^2 \beta_t\, p \ , \tag{144a}
$$

$$
\partial_r M = 4\pi r^2 \beta_r\, \rho \ . \tag{144b}
$$

Equations (144) amply justify the interpretation of M as the interior mass. The first equation (144a) can be written

$$
\boxed{\partial_t M + p\,4\pi r^2 \partial_t r = 0} \tag{145}
$$

which can be recognized as an expression of the first law of thermodynamics

$$dE + p\,dV = 0 \tag{146}$$

with mass-energy E equal to M. The second equation (144b) can be written, since $\partial_r = \beta_r\,\partial/\partial r$, equation (113),

$$\boxed{\frac{\partial M}{\partial r} = 4\pi r^2 \rho} \tag{147}$$

which looks exactly like the Newtonian relation between interior mass M and density ρ. Actually, this apparently Newtonian equation (147) is a bit deceiving. The proper 3-volume element d^3r in the center-of-mass frame is given by (in a notation that is not yet familiar, but clearly has a high class pedigree)

$$d^3r\ \boldsymbol{\gamma}_r \wedge \boldsymbol{\gamma}_\theta \wedge \boldsymbol{\gamma}_\phi = \boldsymbol{g}_r\,dr \wedge \boldsymbol{g}_\theta\,d\theta \wedge \boldsymbol{g}_\phi\,d\phi = \frac{r^2 \sin\theta\,dr\,d\theta\,d\phi}{\beta_r}\,\boldsymbol{\gamma}_r \wedge \boldsymbol{\gamma}_\theta \wedge \boldsymbol{\gamma}_\phi \tag{148}$$

so that the proper 3-volume element dV of a radial shell of width dr is

$$dV = \frac{4\pi r^2 dr}{\beta_r}\ . \tag{149}$$

Thus the "true" mass-energy dM_m associated with the proper density ρ in a proper radial volume element dV might be expected to be

$$dM_m = \rho\,dV = \frac{4\pi r^2 dr}{\beta_r} \tag{150}$$

whereas equation (147) indicates that the actual mass-energy is

$$dM = \rho\,4\pi r^2 dr = \beta_r\,\rho\,dV\ . \tag{151}$$

A person in the center-of-mass frame might perhaps, although there is really no formal justification for doing so, interpret the balance of the mass-energy as gravitational mass-energy M_g

$$dM_g = (\beta_r - 1)\rho\,dV\ . \tag{152}$$

Whatever the case, the moral of this is that you should beware of interpreting the interior mass M too literally as palpable mass-energy.

5.23.10 Energy-momentum conservation

Covariant conservation of the Einstein tensor $D_m G^{mn} = 0$ implies energy-momentum conservation $D_m T^{mn} = 0$. The two non-vanishing equations represent conservation of energy and of radial momentum, and are

$$\boxed{D_m T^{mt} = \partial_t \rho + \frac{2\beta_t}{r}(\rho + p_\perp) + h\,(\rho + p) + \Big(\partial_r + \frac{2\beta_r}{r} + 2\,g\Big) f = 0} \ , \tag{153a}$$

$$\boxed{D_m T^{mr} = \partial_r p + \frac{2\beta_r}{r}(p - p_\perp) + g\,(\rho + p) + \Big(\partial_t + \frac{2\beta_t}{r} + 2\,h\Big) f = 0} \ . \tag{153b}$$

In the center-of-mass frame, $f = 0$, these energy-momentum conservation equations reduce to

$$\partial_t \rho + \frac{2\beta_t}{r}(\rho + p_\perp) + h\,(\rho + p) \ = \ 0 \ , \tag{154a}$$

$$\partial_r p + \frac{2\beta_r}{r}(p - p_\perp) + g\,(\rho + p) \ = \ 0 \ . \tag{154b}$$

In a general situation where the mass-energy is the sum over several individual components a,

$$T^{mn} = \sum_{\text{species } a} T_a^{mn} \ , \tag{155}$$

the individual mass-energy components a of the system each satisfy an energy-momentum conservation equation of the form

$$D_m T_a^{mn} = F_a^n \tag{156}$$

where F_a^n is the flux of energy into component a. Einstein's equations enforce energy-momentum conservation of the system as a whole, so the sum of the energy fluxes must be zero

$$\sum_{\text{species } a} F_a^n = 0 \ . \tag{157}$$

5.23.11 First law of thermodynamics

For an individual species a, the energy conservation equation (153a) in the center-of-mass frame of the species can be written

$$D_m T_a^{mt} = \partial_t \rho_a + (\rho_a + p_{\perp a}) \partial_t \ln r^2 + (\rho_a + p_a) \partial_t \ln \lambda_a = F_a^t \tag{158}$$

where λ_a is the radial "scale factor", equation (132), in the center-of-mass frame of the species (the scale factor is different in different frames). Equation (158) can be recognized as an expression of the first law of thermodynamics for a volume element V of species a, in the form

$$V^{-1} \Big[\partial_t (\rho_a V) + p_{\perp a}\, V_r\, \partial_t V_\perp + p_a\, V_\perp\, \partial_t V_r \Big] = F_a^t \tag{159}$$

with transverse volume (area) $V_\perp \propto r^2$, radial volume (width) $V_r \propto \lambda_a$, and total volume $V \propto V_\perp V_r$. The flux F_a^t on the right hand side is the heat per unit volume per unit time going into species a. If the pressure of species a is isotropic, $p_{\perp a} = p_a$, then equation (159) simplifies to

$$V^{-1}\Big[\partial_t(\rho_a V) + p_a\,\partial_t V\Big] = F_a^t \tag{160}$$

with volume $V \propto r^2\lambda_a$.

5.23.12 Structure of the Einstein equations

The spherically symmetric spacetime under consideration is described by 3 vierbein (or metric) coefficients, α, β_t, and β_r. However, some combination of the 3 coefficients represents a gauge freedom, since the spherically symmetric spacetime has only two physical degrees of freedom. As commented in §5.23.8, various gauge-fixing choices can be made, such as choosing to work in the center-of-mass frame, $f = 0$.

Equations (137) give 5 equations for the 4 non-vanishing components of the Einstein tensor in terms of the vierbein coefficients, but only 4 of the equations are independent, since the 2 equations for G^{tr} are equivalent by the definitions (128) and (129) of g and h. Conservation of energy-momentum of the system as a whole is built in to the Einstein equations, a consequence of the Bianchi identities, so 2 of the Einstein equations are effectively equivalent to the energy-momentum conservation equations (153). In the general case where the matter contains multiple components, it is usually a good idea to include the equations describing the conservation or exchange of energy-momentum separately for each component, so that global conservation of energy-momentum is then satisfied as a consequence of the matter equations.

This leaves 2 independent Einstein equations to describe the 2 physical degrees of the spacetime. The 2 equations may be taken to be the evolution equations (137a) and (137d) for β_t and β_r

$$\boxed{D_t\beta_t = \partial_t\beta_t - \beta_r g = -\frac{M}{r^2} - 4\pi r p}\ , \tag{161a}$$

$$\boxed{D_t\beta_r = \partial_t\beta_r - \beta_t g = 4\pi r f}\ , \tag{161b}$$

which are valid for any choice of tetrad frame, not just the center-of-mass frame.

Equation (161a) is perhaps the single most important of the general relativistic equations governing spherically symmetric spacetimes, because it is this equation that is responsible (to the extent that equations may be considered responsible) for the strange internal structure of Reissner-Nordström black holes, and for mass inflation. The coefficient β_t equals the coordinate radial 4-velocity $dr/d\tau = \partial_t r = \beta_t$ of the tetrad frame, equation (114), and thus equation (161a) can be regarded as giving the proper radial acceleration $D^2r/D\tau^2 = D\beta_t/D\tau = D_t\beta_t$ of the tetrad frame as measured by a person who is in free-fall and instantaneously at rest in the tetrad frame. If the acceleration is measured by an observer who is continuously at rest in the tetrad frame (as opposed to being in free-fall), then the proper acceleration is $\partial_t \beta_t$, which contains an extra term $\beta_r g$ compared to $D_t \beta_t$. The presence of this extra term, proportional to the proper acceleration g actually experienced by the observer continuously at rest in the tetrad frame, reflects the principle of equivalence of gravity and acceleration.

The right hand side of equation (161a) can be interpreted as the radial gravitational force, which consists of 2 terms. The first term, $-M/r^2$, looks like the familiar Newtonian gravitational force, which is attractive (negative, inward) in the usual case of positive mass M. But it is the second term, $-4\pi r p$, proportional to the radial pressure p, that is the source of fun. In a Reissner-Nordström black hole, the negative radial pressure produced by the radial electric field produces a radial gravitational repulsion (positive, outward), according to equation (161a), and this repulsion dominates the gravitational force at small radii, producing an inner horizon. Again, in mass inflation, the (positive) radial pressure of relativistically counter-streaming ingoing and outgoing streams just above the inner horizon dominates the gravitational force (inward), and it is this that drives mass inflation.

5.23.13 Comment on the vierbein coefficient α

Whereas the Einstein equations (161) give evolution equations for the vierbein coefficients β_t and β_r, there is no evolution equation for the vierbein coefficient α. Indeed, the Einstein equations involve the vierbein coefficient α only in the combination $g \equiv \partial_r \ln \alpha$. This reflects the fact that, even after the tetrad frame is fixed, there is still a coordinate freedom $t \to t'(t)$ in the choice of coordinate time t. Under such a gauge transformation, α transforms as $\alpha \to \alpha' = f(t)\,\alpha$ where $f(t) = \partial t/\partial t'$ is an arbitrary function of coordinate time t. Only $g \equiv \partial_r \ln \alpha$ is independent of this coordinate gauge freedom, and thus only g appears in the tetrad-frame Einstein equations.

Since α is needed to propagate the equations from one coordinate time to the next [because $\partial_t = (1/\alpha)\,\partial/\partial t + \beta_t\,\partial/\partial r$], it is necessary to construct α by integrating $g \equiv \beta_r\,\partial \ln \alpha/\partial r$ along the radial direction r at each time step. The arbitrary normalization of α at each step might be fixed by choosing α to be unity at infinity, which corresponds to fixing the time coordinate t to equal the proper time at infinity.

In the particular case that the tetrad frame is taken to be in free-fall everywhere, $g = 0$, as in the Gullstrand-Painlevé metric, then α is constant at fixed t, and without loss of generality it can be fixed equal to unity everywhere, $\alpha = 1$. I like to think of a free-fall frame as being realized physically by tracer "dark matter" particles that fall radially (from zero velocity, typically) at infinity, and stream freely, without interacting, through any actual matter that may be present.

5.24 Spherical electromagnetic field

The internal structure of a charged black hole resembles that of a rotating black hole because the negative pressure (tension) of the radial electric field produces a gravitational repulsion analogous to the centrifugal repulsion in a rotating black hole. Since it is much easier to deal with spherical than rotating black holes, it is common to use charge as a surrogate for rotation in exploring black holes.

5.24.1 Electromagnetic field

The assumption of spherical symmetry means that any electromagnetic field can consist only of a radial electric field (in the absence of magnetic monopoles). The only non-vanishing components of the electromagnetic field F_{mn} are then

$$\boxed{-F^{tr} = F^{rt} = E = \frac{Q}{r^2}} \tag{162}$$

where E is the radial electric field, and $Q(t,r)$ is the interior electric charge. Equation (162) can be regarded as defining what is meant by the electric charge Q interior to radius r at time t.

5.24.2 Maxwell's equations

A radial electric field automatically satisfies two of Maxwell's equations, the source-free ones (57). For the radial electric field (162), the other two Maxwell's equations, the sourced ones (58), are

$$\boxed{\partial_r Q \;=\; 4\pi r^2 q} \tag{163a}$$

$$\boxed{\partial_t Q \;=\; -4\pi r^2 j} \tag{163b}$$

where $q \equiv j^t$ is the proper electric charge density and $j \equiv j^r$ is the proper radial electric current density in the tetrad frame.

5.24.3 Electromagnetic energy-momentum tensor

For the radial electric field (162), the electromagnetic energy-momentum tensor (60) in the tetrad frame is the diagonal tensor

$$T_e^{mn} = \frac{Q^2}{8\pi r^4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{164}$$

The radial electric energy-momentum tensor is independent of the radial motion of the tetrad frame, which reflects the fact that the electric field is invariant under a radial Lorentz boost.

The energy density ρ_e and radial and transverse pressures p_e and $p_{\perp e}$ of the electromagnetic field are the same as those from a spherical charge distribution with interior electric charge Q in flat space

$$\rho_e = -p_e = p_{\perp e} = \frac{Q^2}{8\pi r^4} = \frac{E^2}{8\pi} \ . \tag{165}$$

The non-vanishing components of the covariant derivative $D_m T_e^{mn}$ of the electromagnetic energy-momentum (164) are

$$D_m T_e^{mt} = \partial_t \rho_e + \frac{4\beta_t}{r}\rho_e = \quad \frac{Q}{4\pi r^4}\,\partial_t Q = -\,\frac{jQ}{r^2} = -\,jE \ , \tag{166a}$$

$$D_m T_e^{mr} = \partial_r p_e + \frac{4\beta_r}{r}p_e = -\,\frac{Q}{4\pi r^4}\,\partial_r Q = -\,\frac{qQ}{r^2} = -\,qE \ . \tag{166b}$$

The first expression (166a), which gives the rate of energy transfer out of the electromagnetic field as the current density j times the electric field E, is the same as in flat space. The second expression (166b), which gives the rate of transfer of radial momentum out of the electromagnetic field as the charge density q times the electric field E, is the Lorentz force on a charge density q, and again is the same as in flat space.

5.25 General relativistic stellar structure

A star can be well approximated as static as well as spherically symmetric. In this case all time derivatives can be taken to vanish, $\partial/\partial t = 0$, and, since the center-of-mass frame coincides with the rest frame, it is natural to choose the tetrad frame to be at rest, $\beta_t = 0$. Equation (161b) then vanishes identically, while the acceleration equation (161a) becomes

$$\beta_r g = \frac{M}{r^2} + 4\pi r p \ , \tag{167}$$

which expresses the proper acceleration g in the rest frame in terms of the familiar Newtonian gravitational force M/r^2 plus a term $4\pi r p$ proportional to the radial pressure. The radial pressure, if positive as is the usual case for a star, enhances the inward gravitational force, helping to destabilize the star. Because β_t is zero, the interior mass M given by equation (118) reduces to

$$1 - 2M/r = \beta_r^2 \ . \tag{168}$$

When equations (167) and (168) are substituted into the momentum equation (153b), and if the pressure is taken to be isotropic, so $p_\perp = p$, the result is the **Oppenheimer-Volkov equation** for general relativistic hydrostatic equilibrium

$$\boxed{\frac{\partial p}{\partial r} = -\,\frac{(\rho + p)(M + 4\pi r^3 p)}{r^2(1 - 2M/r)}} \ . \tag{169}$$

In the Newtonian limit $p \ll \rho$ and $M \ll r$ this goes over to (with units restored)

$$\frac{\partial p}{\partial r} = -\,\rho\,\frac{GM}{r^2} \ , \tag{170}$$

which is the usual Newtonian equation of spherically symmetric hydrostatic equilibrium.

5.26 Self-similar spherically symmetric spacetime

Even with the assumption of spherical symmetry, it is by no means easy to solve the system of partial differential equations that comprise the Einstein equations coupled to mass-energy of various kinds. One way to simplify the system of equations, transforming them into ordinary differential equations, is to consider self-similar solutions.

5.26.1 Self-similarity

The assumption of **self-similarity** (also known as homothety, if you can pronounce it) is the assumption that the system possesses conformal time translation invariance. This implies that there exists a conformal time coordinate η such that the geometry at any one time is conformally related to the geometry at any other time

$$ds^2 = a(\eta)^2 \left[g^{(\mathrm{c})}_{\eta\eta}(x)\, d\eta^2 + 2\, g^{(\mathrm{c})}_{\eta x}(x)\, d\eta\, dx + g^{(\mathrm{c})}_{xx}(x)\, dx^2 - e^{2x}\, do^2 \right] \ . \tag{171}$$

Here the conformal metric coefficients $g^{(\mathrm{c})}_{\mu\nu}(x)$ are functions only of conformal radius x, not of conformal time η. The choice e^{2x} of coefficient of do^2 is a gauge choice of the conformal radius x, carefully chosen here so as to bring the self-similar metric into a form (176) below that resembles as far as possible the spherical metric (119). In place of the conformal factor $a(\eta)$ it is convenient to work with the circumferential radius r

$$r \equiv a(\eta) e^x \tag{172}$$

which is to be considered as a function $r(\eta, x)$ of the coordinates η and x. The circumferential radius r has a gauge-invariant meaning, whereas neither $a(\eta)$ nor x are independently gauge-invariant. The conformal factor r has the dimensions of length. In self-similar solutions, all quantities are proportional to some power of r, and that power can be determined by dimensional analysis. Quantites that depend only on the conformal radial coordinate x, independent of the circumferential radius r, are called dimensionless.

The fact that dimensionless quantities such as the conformal metric coefficients $g^{(\mathrm{c})}_{\mu\nu}(x)$ are independent of conformal time η implies that the tangent vector $\boldsymbol{g}_\eta$, which by definition satisfies

$$\frac{\partial}{\partial \eta} = \boldsymbol{g}_\eta \cdot \boldsymbol{\partial} \ , \tag{173}$$

is a **conformal Killing vector**, also known as the homothetic vector. The tetrad-frame components of the conformal Killing vector $\boldsymbol{g}_\eta$ defines the tetrad-frame conformal Killing 4-vector ξ^m

$$\boxed{\frac{\partial}{\partial \eta} \equiv r\, \xi^m \partial_m} \ , \tag{174}$$

in which the factor r is introduced so as to make ξ^m dimensionless. The conformal Killing vector $\boldsymbol{g}_\eta$ is the generator of the conformal time translation symmetry, and as such it is gauge-invariant (up to a global rescaling of conformal time, $\eta \to b\eta$ for some constant b). It follows that its dimensionless tetrad-frame components ξ^m constitute a tetrad 4-vector (again, up to global rescaling of conformal time).

5.26.2 Vierbein

The self-similar vierbein ${e_m}^\mu$ and its inverse ${e^m}_\mu$ can be taken to be of the same form as before, equations (112), but it is convenient to make the dependence on the dimensionless conformal Killing vector ξ^m manifest:

$$\boxed{{e_m}^\mu = \frac{1}{r}\begin{pmatrix} 1/\xi^\eta & -\beta_x\,\xi^x/\xi^\eta & 0 & 0 \\ 0 & \beta_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/\sin\theta \end{pmatrix} \ , \quad {e^m}_\mu = r\begin{pmatrix} \xi^\eta & 0 & 0 & 0 \\ \xi^x & 1/\beta_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sin\theta \end{pmatrix}} \ . \tag{175}$$

It is straightforward to see that the coordinate time components of the inverse vierbein must be ${e^m}_\eta = r\,\xi^m$, since $\partial/\partial\eta = {e^m}_\eta\,\partial_m$ equals $r\,\xi^m\partial_m$, equation (174).

5.26.3 Coordinate metric

The coordinate metric $ds^2 = \eta_{mn}{e^m}_\mu {e^n}_\nu dx^\mu dx^\nu$ corresponding to the vierbein (175) is

$$ds^2 = r^2\left[(\xi^\eta\,d\eta)^2 - \frac{1}{\beta_x^2}\,(dx + \beta_x\,\xi^x d\eta)^2 - do^2\right] \ . \tag{176}$$

5.26.4 Tetrad-frame scalars and vectors

Since the conformal factor r is gauge-invariant, the directed gradient $\partial_m r$ constitutes a tetrad-frame 4-vector β_m (which unlike ξ^m is independent of any global rescaling of conformal time)

$$\boxed{\beta_m \equiv \partial_m r} \ . \tag{177}$$

It is straightforward to check that β_x defined by equation (177) is consistent with its appearance in the vierbein (175) provided that $r \propto e^x$ as earlier assumed, equation (172).

With two distinct dimensionless tetrad 4-vectors in hand, β_m and the conformal Killing vector ξ^m, three gauge-invariant dimensionless scalars can be constructed, $\beta^m\beta_m$, $\xi^m\beta_m$, and $\xi^m\xi_m$,

$$\boxed{1 - \frac{2M}{r} = \beta^m\beta_m = -\,\beta_\eta^2 + \beta_x^2} \ , \tag{178}$$

$$\boxed{v \equiv \xi^m\beta_m = \frac{1}{r}\frac{\partial r}{\partial\eta} = \frac{1}{a}\frac{\partial a}{\partial\eta}} \ , \tag{179}$$

$$\boxed{\Delta \equiv \xi^m\xi_m = (\xi^\eta)^2 - (\xi^x)^2} \ . \tag{180}$$

Equation (178) is essentially the same as equation (118).

The dimensionless quantity v, equation (179), may be interpreted as a measure of the expansion velocity of the self-similar spacetime. Equation (179) shows that v is a function only of η (since $a(\eta)$ is a function only of η), and it therefore follows that v must be constant (since being dimensionless means that v must be a function only of x). Equation (179) then also implies that the conformal factor $a(\eta)$ must take the form

$$a(\eta) = e^{\mathrm{v}\eta} . \tag{181}$$

Because of the freedom of a global rescaling of conformal time, it is possible to set $\mathrm{v} = 1$ without loss of generality, but in practice it is convenient to keep v, because it is then transparent how to take the static limit $\mathrm{v} \to 0$. Equation (181) along with (172) shows that the circumferentaial radius r is related to the conformal coordinates η and x by

$$\boxed{r = e^{\mathrm{v}\eta + x}} . \tag{182}$$

The dimensionless quantity Δ, equation (180), is the dimensionless horizon function: horizons occur where the horizon function vanishes

$$\boxed{\Delta = 0} \quad \text{at horizons} . \tag{183}$$

5.26.5 Diagonal coordinate metric of the similarity frame

The metric (176) can be brought to diagonal form by a coordinate transformation to diagonal conformal coordinates $\eta_\times$, $x_\times$ (subscripted $\times$ for diagonal)

$$\eta \to \eta_\times = \eta + f(x) , \quad x \to x_\times = x - \mathrm{v} f(x) , \tag{184}$$

which leaves unchanged the conformal factor r, equation (182). The resulting diagonal metric is

$$ds^2 = r^2 \left(\Delta \, d\eta_\times^2 - \frac{dx_\times^2}{1 - 2M/r + \mathrm{v}^2/\Delta} - do^2 \right) . \tag{185}$$

The diagonal metric (185) corresponds physically to the case where the tetrad frame is at rest in the similarity frame, $\xi^x = 0$, as can be seen by comparing it to the metric (176). The frame can be called the **similarity frame**. The form of the metric coefficients follows from the metric (176) and the gauge-invariant scalars (178)–(180).

The conformal Killing vector in the similarity frame is $\xi^m = \{\Delta^{1/2}, 0, 0, 0\}$, and the 4-velocity of the similarity frame in its own frame is $u^m = \{1, 0, 0, 0\}$. Since both are tetrad 4-vectors, it follows that with respect to a general tetrad frame

$$\xi^m = u^m \, \Delta^{1/2} \tag{186}$$

where u^m is the 4-velocity of the similarity frame with respect to the general frame. This shows that the conformal Killing vector ξ^m in a general tetrad frame is proportional to the 4-velocity of the similarity frame through the tetrad frame. In particular, the proper 3-velocity of the similarity frame through the tetrad frame is

$$\text{proper 3-velocity of similarity frame through tetrad frame} = \frac{\xi^x}{\xi^\eta} . \tag{187}$$

5.26.6 Ray-tracing metric

It proves useful to introduce a "ray-tracing" conformal radial coordinate X related to the coordinate $x_\times$ of the diagonal metric (185) by

$$dX \equiv \frac{\Delta\, dx_\times}{[(1 - 2M/r)\Delta + \mathrm{v}^2]^{1/2}} \ . \tag{188}$$

In terms of the ray-tracing coordinate X, the diagonal metric is

$$ds^2 = r^2 \left(\Delta\, d\eta_\times^2 - \frac{dX^2}{\Delta} - do^2 \right) \ . \tag{189}$$

For the Reissner-Nordström geometry, $\Delta = (1 - 2M/r)/r^2$, $\eta_\times = t$, and $X = -1/r$.

5.26.7 Geodesics

Spherical symmetry and conformal time translation symmetry imply that geodesic motion in spherically symmetric self-similar spacetimes is described by a complete set of integrals of motion.

The integral of motion associated with conformal time translation symmetry can be obtained from Lagrange's equations of motion

$$\frac{d}{d\tau} \frac{\partial \mathcal{L}}{\partial u^\eta} = \frac{\partial \mathcal{L}}{\partial \eta} \tag{190}$$

with effective Lagrangian $\mathcal{L} = g_{\mu\nu} u^\mu u^\nu$ for a particle with 4-velocity u^μ. The self-similar metric depends on the conformal time η only through the overall conformal factor $g_{\mu\nu} \propto a^2$. The derivative of the conformal factor is given by $\partial \ln a / \partial \eta = \mathrm{v}$, equation (179), so it follows that $\partial \mathcal{L} / \partial \eta = 2\, \mathrm{v} \mathcal{L}$. For a massive particle, for which conservation of rest mass implies $g_{\mu\nu} u^\mu u^\nu = 1$, Lagrange's equations (190) thus yield

$$\frac{du_\eta}{d\tau} = \mathrm{v} \ . \tag{191}$$

In the limit of zero accretion rate, $\mathrm{v} \to 0$, equation (191) would integrate to give u_η as a constant, the energy per unit mass of the geodesic. But here there is conformal time translation symmetry in place of time translation symmetry, and equation (191) integrates to

$$u_\eta = \mathrm{v}\, \tau \tag{192}$$

in which an arbitrary constant of integration has been absorbed into a shift in the zero point of the proper time τ. Although the above derivation was for a massive particle, it holds also for a massless particle, with the understanding that the proper time τ is constant along a null geodesic. The quantity u_η in equation (192) is the covariant time component of the coordinate-frame 4-velocity u^μ of the particle; it is related to the covariant components u_m of the tetrad-frame 4-velocity of the particle by

$$u_\eta = e^m{}_\eta \, u_m = r \, \xi^m u_m \ . \tag{193}$$

Without loss of generality, geodesic motion can be taken to lie in the equatorial plane $\theta = \pi/2$. The integrals of motion associated with conformal time translation symmetry, rotational symmetry about the polar axis, and conservation of rest mass, are, for a massive particle

$$u_\eta = \text{v}\,\tau \ , \quad u_\phi = -L \ , \quad u_\mu u^\nu = 1 \ , \tag{194}$$

where L is the orbital angular momentum per unit rest mass of the particle. The coordinate 4-velocity $u^\mu \equiv dx^\mu/d\tau$ that follows from equations (194) takes its simplest form in the conformal coordinates $\{\eta_\times, X, \theta, \phi\}$ of the ray-tracing metric (189)

$$u^{\eta_\times} = \frac{\text{v}\,\tau}{r^2 \Delta} \ , \quad u^X = \pm \frac{1}{r^2} \left[\text{v}^2 \tau^2 - (r^2 + L^2) \Delta \right]^{1/2} \ , \quad u^\phi = \frac{L}{r^2} \ . \tag{195}$$

5.26.8 Null geodesics

The important case of a massless particle follows from taking the limit of a massive particle with infinite energy and angular momentum, $\text{v}\,\tau \to \infty$ and $L \to \infty$. To obtain finite results, define an affine parameter λ by $d\lambda \equiv \text{v}\,\tau \, d\tau$, and a 4-velocity in terms of it by $v^\mu \equiv dx^\mu/d\lambda$. The integrals of motion (194) then become, for a null geodesic,

$$v_{\eta_\times} = 1 \ , \quad v_\phi = -J \ , \quad v_\mu v^\nu = 0 \ , \tag{196}$$

where $J \equiv L/(\text{v}\,\tau)$ is the (dimensionless) conformal angular momentum of the particle. The 4-velocity v^μ along the null geodesic is then, in terms of the coordinates of the ray-tracing metric (189),

$$v^\eta = \frac{1}{r^2 \Delta} \ , \quad v^X = \pm \frac{1}{r^2} \left(1 - J^2 \Delta \right)^{1/2} \ , \quad v^\phi = \frac{J}{r^2} \ . \tag{197}$$

Equations (197) yield the shape of a null geodesic by quadrature

$$\phi = \int \frac{J \, dX}{(1 - J^2 \Delta)^{1/2}} \ . \tag{198}$$

Equation (198) shows that the shape of null geodescics in spherically symmetric self-similar spacetimes hinges on the behavior of the dimensionless horizon function $\Delta(X)$ as a function of the dimensionless ray-tracing variable X.

Null geodesics go through periapsis or apoapsis in the self-similar frame where the denominator of the integrand of (198) is zero, corresponding to $v^X = 0$. A photon sphere, where null geodesics circle for ever at constant conformal coordinate X, occurs where the denominator not only vanishes but is an extremum, which happens where the horizon function Δ is an extremum,

$$\frac{d\Delta}{dX} = 0 \quad \text{at photon sphere} \ . \tag{199}$$

5.26.9 Dimensional analysis

Dimensional analysis shows that the conformal coordinates $x^\mu \equiv \{\eta, x, \theta, \phi\}$, the tetrad metric γ_{mn}, and the coordinate metric $g_{\mu\nu}$ are all dimensionless

$$x^\mu \ , \quad \gamma_{mn} \ , \quad g_{\mu\nu} \quad \text{are dimensionless} \ . \tag{200}$$

The vierbein $e_m{}^\mu$ and inverse vierbein $e^m{}_\mu$, equations (175), scale as

$$e_m{}^\mu \propto r^{-1} \ , \quad e^m{}_\mu \propto r \ . \tag{201}$$

Coordinate derivatives $\partial/\partial x^\mu$ are dimensionless, while directed derivatives ∂_m scale as $1/r$

$$\frac{\partial}{\partial x^\mu} \propto r^0 \ , \quad \partial_m \propto r^{-1} \ . \tag{202}$$

The tetrad connections Γ_{kmn} and the tetrad-frame Riemann tensor R_{klmn} scale as

$$\Gamma_{kmn} \propto r^{-1} \ , \quad R_{klmn} \propto r^{-2} \ . \tag{203}$$

5.26.10 Variety of self-similar solutions

Self-similar solutions exist provided that the properties of the energy-momentum introduce no additional dimensional parameters. For example, the pressure-to-density ratio $w \equiv p/\rho$ of any species is dimensionless, and since the ratio can depend only on the nature of the species itself, not for example on where it happens to be located in the spacetime, it follows that the ratio w must be a constant. It is legitimate for the pressure-to-density ratio to be different in the radial and transverse directions (as it is for a radial electric field), but otherwise self-similarity requires that

$$\boxed{w \equiv p/\rho \ , \quad w_\perp \equiv p_\perp/\rho} \ , \tag{204}$$

be constants for each species. For example, $w = 1$ for a massless scalar field, $w = 1/3$ for a relativistic fluid, $w = 0$ for pressureless cold dark matter, $w = -1$ for vacuum energy, and $w = -1$ with $w_\perp = 1$ for a radial electric field.

Self-similarity allows that the energy-momentum may consist of several distinct components, such as a relativistic fluid, plus dark matter, plus an electric field. The components may interact with each other provided that the properties of the interaction introduce no additional dimensional parameters. For example, the relativistic fluid (and the dark matter) may be charged, and if so then the charged fluid will experience a Lorentz force from the electric field, and will therefore exchange momentum with the electric field. If the fluid is non-conducting, then there is no dissipation, and the interaction between the charged fluid and electric field automatically introduces no additional dimensional parameters.

However, if the charged fluid is electrically conducting, then the electrical conductivity could potentially introduce an additional dimensional parameter, and this must not be allowed if self-similarity is to be maintained. In diffusive electrical conduction in a fluid of conductivity σ, an electric field E gives rise to a current

$$\boxed{j = \sigma E} \ , \tag{205}$$

which is just Ohm's law. Dimensional analysis shows that $j \propto r^{-2}$ and $E \propto r^{-1}$, so the conductivity must scale as $\sigma \propto r^{-1}$. The conductivity can depend only on the intrinsic properties of the conducting fluid, and the only intrinsic property available is its density, which scales as $\rho \propto r^{-2}$. If follows that the conductivity must be proportional to the square root of the density ρ of the conducting fluid

$$\boxed{\sigma = \kappa\, \rho^{1/2}} \ , \tag{206}$$

where κ is a dimensionless conductivity constant. The form (206) is required by self-similarity, and is not necessarily realistic (although it is realistic that the conductivity increases with density). However, the conductivity (206) is adequate for the purpose of exploring the consequences of dissipation in simple models of black holes.

5.26.11 Tetrad connections

The expressions for the tetrad connections for the self-similar spacetime are the same as those (127) for a general spherically symmetric spacetime, with just a relabeling of the time and radial coordinates into conformal coordinates

$$t \rightarrow \eta \ , \quad r \rightarrow x \ . \tag{207}$$

Specifically, equations (127) for the tetrad connections become become

$$\Gamma_{\eta x \eta} = g \ , \quad \Gamma_{\eta x x} = h \ , \quad \Gamma_{\eta\theta\theta} = \Gamma_{\eta\phi\phi} = \frac{\beta_\eta}{r} \ , \quad \Gamma_{x\theta\theta} = \Gamma_{x\phi\phi} = \frac{\beta_x}{r} \ , \quad \Gamma_{\theta\phi\phi} = \frac{\cot\theta}{r} \ , \tag{208}$$

in which g and h have the same physical interpretation discussed in §5.23.5 for the general spherically symmetric case: g is the proper radial acceleration, and h is the radial Hubble parameter. Expressions (128) and (129) for g and h translate in the self-similar spacetime to

$$\boxed{g \equiv \partial_x \ln(r\, \xi^\eta) \ , \quad h \equiv \partial_\eta \ln(r\, \xi^x)} \ . \tag{209}$$

Comparing equations (209) to equations (128) and (132) shows that the vierbein coefficent α and scale factor λ translate in the self-similar spacetime to

$$\alpha = r\xi^\eta \ , \quad \lambda = r\xi^x \ . \tag{210}$$

5.26.12 Spherical equations carry over to the self-similar case

The tetrad-frame Riemann, Weyl, and Einstein tensors in the self-similar spacetime take the same form as in the general spherical case, equations (133)–(137), with just a relabeling (207) into conformal coordinates.

Likewise, the equations for the interior mass in §5.23.9, for energy-momentum conservation in §5.23.10, for the first law in §5.23.11, and the various equations for the electromagnetic field in §5.24, all carry through unchanged except for a relabeling (207) of coordinates.

5.26.13 From partial to ordinary differential equations

The central simplifying feature of self-similar solutions is that they turn a system of partial differential equations into a system of ordinary differential equations.

By definition, a dimensionless quantity $F(x)$ is independent of conformal time η. It follows that the partial derivative of any dimensionless quantity $F(x)$ with respect to conformal time η vanishes

$$\boxed{0 = \frac{\partial F(x)}{\partial \eta} = \xi^m \partial_m F(x) = \left(\xi^\eta \partial_\eta + \xi^x \partial_x\right) F(x)} \ . \tag{211}$$

Consequently the directed radial derivative $\partial_x f$ of a dimensionless quantity $F(x)$ is related to its directed time derivative $\partial_\eta f$ by

$$\partial_x F(x) = -\frac{\xi^x}{\xi^\eta} \partial_\eta F(x) \ . \tag{212}$$

Equation (212) allows radial derivatives to be converted to time derivatives.

5.26.14 Integrals of motion

As remarked above, equation (211), in self-similar solutions $\xi^m \partial_m F(x) = 0$ for any dimensionless function $F(x)$. If both the directed derivatives $\partial_\eta F(x)$ and $\partial_x F(x)$ are known from the Einstein equations or elsewhere, then the result will be an integral of motion.

The spherically symmetric, self-similar Einstein equations admit two integrals of motion

$$0 \ = \ r\,\xi^m \partial_m \beta_\eta \ = \ r\,\beta_x(\xi^\eta g + \xi^x h) - \xi^\eta \left(\frac{M}{r} + 4\pi r^2 p\right) + \xi^x 4\pi r f \ , \tag{213a}$$

$$0 \ = \ r\,\xi^m \partial_m \beta_x \ = \ r\,\beta_\eta(\xi^\eta g + \xi^x h) + \xi^x \left(\frac{M}{r} - 4\pi r^2 \rho\right) + \xi^\eta 4\pi r f \ . \tag{213b}$$

In the center-of-mass frame, $f = 0$, these integrals of motion simplify to

$$0 \ = \ r\,\xi^m \partial_m \beta_\eta \ = \ r\,\beta_x(\xi^\eta g + \xi^x h) - \xi^\eta \left(\frac{M}{r} + 4\pi r^2 p\right) \ , \tag{214a}$$

$$0 \ = \ r\,\xi^m \partial_m \beta_x \ = \ r\,\beta_\eta(\xi^\eta g + \xi^x h) + \xi^x \left(\frac{M}{r} - 4\pi r^2 \rho\right) \ . \tag{214b}$$

Taking β_η times (214a) minus β_x times (214b) gives, in the center-of-mass frame,

$$0 = r\,\xi^m \partial_m \frac{M}{r} = -\,\mathrm{v}\,\frac{M}{r} + 4\pi r^2 \left(\xi^x \beta_x \rho - \xi^\eta \beta_\eta p\right) \ . \tag{215}$$

For electrically charged solutions, a third integral of motion comes from

$$0 = r\,\xi^m \partial_m \frac{Q}{r} = -\,\mathrm{v}\,\frac{Q}{r} + 4\pi r^2 \left(\xi^x q - \xi^\eta j\right) \tag{216}$$

which is valid in any radial tetrad frame, not just the center-of-mass frame.

For a fluid with equation of state $p = w\rho$, a fourth integral comes from considering

$$0 = r\,\xi^m \partial_m (r^2 p) = r\left[w\,\xi^\eta \partial_\eta (r^2 \rho) + \xi^x \partial_x (r^2 p)\right] \tag{217}$$

and simplifying using the energy conservation equation for $\partial_\eta \rho$ and the momentum conservation equation for $\partial_x p$.

5.26.15 Integration variable

It is desirable to choose an integration variable that varies monotonically. A natural choice is the proper time τ of the baryonic fluid, since this is guaranteed to increase monotonically. Since the 4-velocity at rest in the tetrad frame is by definition $u^m = \{1, 0, 0, 0\}$, the proper time derivative is related to the directed conformal time derivative in the baryonic tetrad frame by $d/d\tau = u^m \partial_m = \partial_\eta$.

However, there is another choice of integration variable, the ray-tracing variable X defined by equation (188), that is not specifically tied to the tetrad frame of the baryons, and that has a desirable (tetrad and coordinate) gauge-invariant meaning. The proper time derivative of any dimensionless function $F(x)$ in the tetrad frame is related to its derivative dF/dX with respect to the ray-tracing variable X by

$$\partial_\eta F = u^m \partial_m F = u^X \partial_X F = -\frac{\xi^x}{r}\frac{dF}{dX}\ . \tag{218}$$

In the third expression, $u^X \partial_X F$ is $u^m \partial_m F$ expressed in the similarity frame of §5.26.5, the time contribution $u^{\eta_\times} \partial_{\eta_\times} F$ vanishing in the similarity frame because it is proportional to the conformal time derivative $\partial F/\partial \eta_\times = 0$. In the last expression of (218), u^X has been replaced by $-u^x = -\xi^x/\Delta^{1/2}$ in view of equation (186), the minus sign coming from the fact that u^X is the radial component of the tetrad 4-velocity of the tetrad frame relative to the similarity frame, while u^x in equation (186) is the radial component of the tetrad 4-velocity of the similarity frame relative to the tetrad frame. Also in the last expression of (218), the directed derivative ∂_X with respect to the ray-tracing variable X has been translated into its coordinate partial derivative, $\partial_X = (\Delta^{1/2}/r)\,\partial/\partial X$, which follows from the metric (189).

In summary, the chosen integration variable is the dimensionless ray-tracing variable $-X$ (with a minus because $-X$ is monotonically increasing), the derivative with respect to which, acting on any dimensionless function, is related to the proper time derivative in any tetrad frame (not just the baryonic frame) by

$$\boxed{-\frac{d}{dX} = \frac{r}{\xi^x}\,\partial_\eta}\ . \tag{219}$$

Equation (219) involves ξ^x, which is proportional to the proper velocity of the tetrad frame through the similarity frame, equation (187), and which therefore, being initially positive, must always remain positive as long as the fluid does not turn back on itself, as must be true for the self-similar solution to be consistent.

5.26.16 Summary of equations for a single charged fluid

For reference, it is helpful to collect here the full set of equations governing self-similar spherically symmetric evolution in the case of a single charged "baryonic" fluid (hereafter subscripted b) with isotropic equation of state

$$\boxed{p_b = p_{\perp b} = w\,\rho_b} \ , \tag{220}$$

and conductivity

$$\boxed{\sigma_b = \kappa_b\,\rho_b^{1/2}} \ . \tag{221}$$

In accordance with the arguments in §5.26.10, equations (204) and (206), self-similarity requires that the pressure-to-density ratio w_b and the conductivity coefficent κ_b both be (dimensionless) constants.

It is natural to work in the center-of-mass frame of the baryonic fluid, which also coincides with the center-of-mass frame of the fluid plus electric field (the electric field, being invariant under Lorentz boosts, does not pick out any particular radial frame).

The proper time τ in the baryonic frame evolves as

$$\boxed{-\frac{d\tau}{dX} = \frac{r}{\xi^x}} \ , \tag{222}$$

which follows from equation (219) and the fact that $\partial_\eta \tau = 1$. The circumferential radius r evolves along the path of the baryonic fluid as

$$\boxed{-\frac{d\ln r}{dX} = \frac{\beta_\eta}{\xi^x}} \ . \tag{223}$$

Although it is straightforward to write down the equations governing how the baryonic tetrad frame moves through the conformal coordinates η and x, there is not much to be gained from this because the conformal coordinates have no fundamental physical significance.

Next, the defining equations (209) for the proper acceleration g and Hubble parameter h yield equations for the evolution of the time and radial components of the conformal Killing vector ξ^m

$$\boxed{-\frac{d\xi^\eta}{dX} = \beta_x - rg} \ , \tag{224a}$$

$$\boxed{-\frac{d\xi^x}{dX} = -\beta_\eta + rh} \ , \tag{224b}$$

in which, in the formula for g, equation (212) has been used to convert the conformal radial derivative ∂_x to the conformal time derivative ∂_η, and thence to $-d/dX$ by equation (219).

Next, the Einstein equations (137b) and (137a) [with coordinates relabeled per (207] in the center-of-mass frame (142) yield evolution equations for the time and radial components of the vierbein coefficients β_m

$$\boxed{-\frac{d\beta_\eta}{dX} \ = \ -\frac{\beta_x}{\xi^\eta}\,rh} \ , \tag{225a}$$

$$\boxed{-\frac{d\beta_x}{dX} \ = \ \frac{\beta_\eta}{\xi^x}\,rg} \ , \tag{225b}$$

where again, in the formula for β_η, equation (212) has been used to convert the conformal radial derivative ∂_x to the conformal time derivative ∂_η. The 4 evolution equations (224) and (225) for ξ^m and β_m are not independent: they are related by $\xi^m\beta_m = \mathrm{v}$, a constant, equation (179). To maintain numerical precision, it is important to avoid expressing small quantities as differences of large quantities. In practice, a suitable choice of variables to integrate proves to be $\xi^\eta{+}\xi^x$, $\beta_\eta{-}\beta_x$, and β_x, each of which can be tiny in some circumstances. Starting from these variables, the following equations yield $\xi^\eta - \xi^x$, along with the interior mass M and the horizon function Δ, equations (178) and (180), in a fashion that ensures numerical stability:

$$\boxed{\xi^\eta - \xi^x \ = \ \frac{2\mathrm{v} - (\xi^\eta + \xi^x)(\beta_\eta + \beta_x)}{\beta_\eta - \beta_x}} \ , \tag{226a}$$

$$\boxed{\frac{2M}{r} \ = \ 1 + (\beta_\eta + \beta_x)(\beta_\eta - \beta_x)} \ , \tag{226b}$$

$$\boxed{\Delta \ = \ (\xi^\eta + \xi^x)(\xi^\eta - \xi^x)} \ . \tag{226c}$$

The evolution equations (224) and (225) involve g and h. The integrals of motion considered in §5.26.14 yield explicit expressions for g and h not involving any derivatives. For the Hubble parameter h, taking ξ^x times the integral of motion (214a) plus ξ^η times (214b) yields

$$\boxed{rh = -\frac{\xi^\eta}{\xi^x}\,rg + \frac{\xi^\eta}{\mathrm{v}}\,4\pi\varepsilon} \ , \tag{227}$$

where ε is the enthalpy

$$\boxed{\varepsilon \equiv \rho + p = (1 + w_b)\rho_b} \ , \tag{228}$$

in which the last equality is true because the electromagnetic enthalpy is identically zero, $\rho_e + p_e = 0$, equation (165). For the proper acceleration g, a somewhat lengthy calculation starting from the integral of motion (217), and simplifying using the integrals of motion (215) for M and (216) for Q, the expression (227) for h, Maxwell's equation (163b) [with the relabeling (207)], and the conductivity (221) in Ohm's law (205), gives

$$\boxed{rg = \frac{\xi^x\left\{2w_b\mathrm{v}M/r + \left[(1-w_b)\mathrm{v} + (1+w_b)4\pi r\sigma_b\xi^\eta\right]Q^2/r^2 - w_b(4\pi\xi^\eta\varepsilon)^2/\mathrm{v}\right\}}{4\pi\varepsilon\left[(\xi^x)^2 - w_b(\xi^\eta)^2\right]}} \ . \tag{229}$$

Two more equations complete the suite. The first, which represents energy conservation for the baryonic fluid, can be written as an equation governing the entropy S_b of the fluid

$$\boxed{-\frac{d\ln S_b}{dX} = \frac{\sigma_b Q^2}{r(1+w_b)\rho_b \xi^x}} \ , \tag{230}$$

in which the S_b is (up to an arbitrary constant) the entropy of a comoving volume element $V \propto r^3 \xi^x$ of the baryonic fluid

$$\boxed{S_b \equiv r^3 \xi^x \rho_b^{1/(1+w_b)}} \ . \tag{231}$$

That equation (230) really is an entropy equation can be confirmed by rewriting it as

$$\frac{1}{V}\left(\frac{d\rho_b V}{d\tau} + p_b \frac{dV}{d\tau}\right) = jE = \frac{\sigma_b Q^2}{r^4} \ , \tag{232}$$

in which jE is recognized as the Ohmic dissipation, the rate per unit volume at which the baryonic volume element V is being heated.

The final equation represents electromagnetic energy conservation, equation (166a), which can be written

$$\boxed{-\frac{d\ln Q}{dX} = -\frac{4\pi r \sigma_b}{\xi^x}} \ . \tag{233}$$

The (heat) energy going into the baryonic fluid is balanced by the (free) energy coming out of the electromagnetic field.

5.26.17 Messenger from the outside universe

In the Reissner-Nordström (and Kerr-Newman) geometries, a person passing through the outgoing inner horizon sees the entire future of the outside universe go by in an infinitely blueshifted flash. Violent things happen also to a person who falls into a realistic black hole, but do those violent things depend only on what happens in the infinite future? If so, then it makes the predictions less credible, because a lot can happen in the infinite future, such as mergers of the black hole with other black holes, evaporation of the black hole, and unfathomables beyond our ken.

In practice, the computations show that the extreme things that happen inside black holes do not depend on what happens in the distant future. On the contrary, practically no time goes by in the outside universe. To check that this is the case, it is convenient to introduce a messenger from the outside universe, in the form of radially free-falling non-interacting pressureless tracer dark matter (subscripted d), which can be taken to be either massless (hot) or massive (cold).

By assumption, the messenger dark matter is freely-falling along a radial geodesic. If the dark matter is massive, then the 4-velocity of the dark matter in its own frame is by definition $u_d^m = \{1, 0, 0, 0\}$, and it follows from the integral of motion (192) coupled with the expression (193) that

$$u_{d,\eta} = r_d \, \xi_d^\eta = v \, \tau_d \tag{234}$$

where r_d is the circumferential radius along the geodesic, and τ_d is the proper time attached to the dark matter particle. Equation (234) can be taken to be true also for a massless dark matter particle, on the understanding that, upon rescaling to the affine parameter, the 4-vectors u_d^m, ξ_d^m, and $\beta_{d,m}$ all become null 4-vectors.

The proper time τ_d attached to the freely-falling dark matter particles provides a clock that tells the baryonic fluid inside the black hole how much time has passed in the outside universe. During mass inflation, the baryonic fluid may see the dark matter as extremely highly blueshifted, but whether that high blueshift translates into a lot of time going by in the outside universe can be checked by looking at the dark matter clock.

One application of the dark matter clock, which will be used in §5.27, is to determine the accretion rate $\dot{M}_\bullet$ of the black hole as seen from afar. It is not satisfactory to measure the accretion rate with respect to time recorded on clocks near the black hole, because that time could differ substantially from that measured at infinity. If the infalling dark matter clocks are synchronized to the proper time on clocks at rest at infinity, then the time on successive dark matter clocks falling through any fiducial point will represent the time at rest at infinity.

Equation (234) involves three unknowns, the circumferential radius r_d along the path of the dark matter (which evolves differently from the circumferential radius r in another frame, such as the baryonic frame), the proper time τ_d of the dark matter, and the time component ξ_d^η of the conformal Killing vector in the dark matter frame. Solving for two of these will yield the third. Analogously to equations (222) and (223), the equations governing the evolution of τ_d and r_d are, with $\mu = 1$ for massive, $\mu = 0$ for massless dark matter,

$$-\frac{d\tau_d}{dX} = \frac{\mu^2 r_d}{\xi_d^x} \ , \tag{235}$$

$$-\frac{d\ln r_d}{dX} = \frac{\beta_{d,\eta}}{\xi_d^x} \ . \tag{236}$$

Given these, the conformal Killing vector ξ_d^m in the dark matter frame, the 4-velocity u_d^m of the dark matter relative to the baryonic frame, and the radial 4-gradient $\beta_{d,m}$ in the dark matter frame follow according to the following chain of equations, which are organized so as to ensure numerical accuracy:

$$\xi_d^\eta = \frac{\mathrm{v}\,\tau_d}{r_d} \ , \tag{237a}$$

$$\xi_d^x = \left[(\xi_d^\eta)^2 - \mu^2\Delta\right]^{1/2} \ , \tag{237b}$$

$$u_d^\eta - u_d^x = \frac{\xi_d^\eta + \xi_d^x}{\xi^\eta + \xi^x} \ , \tag{237c}$$

$$u_d^\eta + u_d^x = \frac{\mu^2}{u_d^\eta - u_d^x} \ , \tag{237d}$$

$$\beta_{d,\eta} \pm \beta_{d,x} = (\beta_\eta \pm \beta_x)\,(u_d^\eta \pm u_d^x) \ . \tag{237e}$$

5.27 Self-similar models of the interior structure of black holes

The apparatus is now in hand actually to do some real calculations of the interior structure of black holes. All the models presented in this section are spherical and self-similar. See Hamilton & Pollack (2005, PRD 71, 084031 & 084031) and Wallace, Hamilton & Polhemus (2008, arXiv:0801.4415) for more.

5.27.1 Boundary conditions at an outer sonic point

Because information can propagate only inward inside the horizon of a black hole, it is natural to set the boundary conditions outside the horizon. The policy adopted here is to set them at a sonic point, where the infalling fluid accelerates from subsonic to supersonic. The proper 3-velocity of the fluid through the self-similar frame is ξ^x/ξ^η, equation (187) (the velocity ξ^x/ξ^η is positive falling inward), and the sound speed is

$$\text{sound speed} = \sqrt{\frac{p_b}{\rho_b}} = \sqrt{w_b} \ , \tag{238}$$

and sonic points occur where the velocity equals the sound speed

$$\frac{\xi^x}{\xi^\eta} = \pm\sqrt{w_b} \quad \text{at sonic points} \ . \tag{239}$$

The denominator of the expression (229) for the proper acceleration g is zero at sonic points, indicating that the acceleration will diverge unless the numerator is also zero. What happens at a sonic point depends on whether the fluid transitions from subsonic upstream to supersonic downstream (as here) or vice versa. If (as here) the fluid transitions from subsonic to supersonic, then sound waves generated by discontinuities near the sonic point can propagate upstream, plausibly modifying the flow so as to ensure a smooth transition through the sonic point, effectively forcing the numerator, like the denominator, of the expression (229) to pass through zero at the sonic point. Conversely, if the fluid transitions from supersonic to subsonic, then sound waves cannot propagate upstream to warn the incoming fluid that a divergent acceleration is coming, and the result is a shock wave, where the fluid accelerates discontinuously, is heated, and thereby passes from supersonic to subsonic.

The solutions considered here assume that the acceleration g at the sonic point is not only continuous [so the numerator of (229) is zero] but also differentiable. Such a sonic point is said to be regular, and the assumption imposes two boundary conditions at the sonic point.

The accretion in real black holes is likely to be much more complicated, but the assumption of a regular sonic point is the simplest physically reasonable one.

5.27.2 Mass and charge of the black hole

The mass $M_\bullet$ and charge $Q_\bullet$ of the black hole at any instant can be defined to be those that would be measured by a distant observer if there were no mass or charge outside the sonic point

$$M_\bullet = M + \frac{Q^2}{2r} \ , \quad Q_\bullet = Q \quad \text{at the sonic point} \ . \tag{240}$$

The mass $M_\bullet$ in equation (240) includes the mass-energy $Q^2/2r$ that would be in the electric field outside the sonic point if there were no charge outside the sonic point, but it does not include mass-energy from any additional mass or charge that might be outside the sonic point.

In self-similar evolution, the black hole mass increases linearly with time, $M_\bullet \propto t$, where t is the proper time at rest far from the black hole. As discussed in §5.26.17, this time t equals the proper time $\tau_d = r\xi_d^\eta/v$ recorded by dark matter clocks that free-fall radially from zero velocity at infinity. Thus the mass accretion rate $\dot{M}_\bullet$ is

$$\dot{M}_\bullet \equiv \frac{dM_\bullet}{dt} = \frac{M_\bullet}{\tau_d} = \frac{vM_\bullet}{r\xi_d^\eta} \quad \text{at the sonic point} \ . \tag{241}$$

If there is no mass outside the sonic point (apart from the mass-energy in the electric field), then a freely-falling dark matter particle will have

$$\beta_{d,x} = 1 \quad \text{at the sonic point} \ , \tag{242}$$

which can be taken as the boundary condition on the dark matter at the sonic point, for either massive or massless dark matter. Equation (242) follows from the facts that the geodesic equations in empty space around a charged black hole (Reissner-Nordström metric) imply that $\beta_{d,x} =$ constant for a radially free-falling particle (the same conclusion can drawn from the Einstein equation (137a)), and that a particle at rest at infinity satisfies $\beta_{d,\eta} = \partial_{d,\eta} r = 0$, and consequently $\beta_{d,x} = 1$ from equation (178) with $r \to \infty$.

As remarked following equation (181), the residual gauge freedom in the global rescaling of conformal time η allows the expansion velocity v to be adjusted at will. One choice suggested by equation (241) is to set (but one could equally well set v to the expansion velocity of the horizon, $v = \dot{r}_+$, for example)

$$v = \dot{M}_\bullet \ , \tag{243}$$

which is equivalent to setting

$$\xi_d^\eta = \frac{M_\bullet}{r} \quad \text{at the sonic point} \ . \tag{244}$$

Equation (244) is not a boundary condition: it is just a choice of units of conformal time η. Equation (244) and the boundary condition (242) coupled with the scalar relations (178) and (179) fully determine the dark matter 4-vectors $\beta_{d,m}$ and ξ_d^m at the sonic point.

5.27.3 Equation of state

The density ρ_b and temperature T_b of an ideal relativistic baryonic fluid in thermodynamic equilibrium are related by

$$\rho_b = \frac{\pi^2 g}{30} T_b^4 , \tag{245}$$

where

$$g = g_B + \frac{7}{8} g_F \tag{246}$$

is the effective number of relativistic particle species, with g_B and g_F being the number of bosonic and fermionic species. If the expected increase in g with temperature T is modeled (so as not to spoil self-similarity) as a weak power law $g/g_{\rm p} = T^\epsilon$, with $g_{\rm p}$ the effective number of relativistic species at the Planck temperature, then the relation between density ρ_b and temperature T_b is

$$\rho_b = \frac{\pi^2 g_{\rm p}}{30} T_b^{(1+w)/w} , \tag{247}$$

with equation of state parameter $w_b = 1/(3 + \epsilon)$ slightly less than the standard relativistic value $w = 1/3$. In the models considered here, the baryonic equation of state is taken to be

$$w_b = 0.32 . \tag{248}$$

The effective number g_p is fixed by setting the number of relativistic particles species to $g = 5.5$ at $T = 10\,\mathrm{MeV}$, corresponding to a plasma of relativistic photons, electrons, and positrons. This corresponds to choosing the effective number of relativistic species at the Planck temperature to be $g_{\rm p} \approx 2{,}400$, which is not unreasonable.

The chemical potential of the relativistic baryonic fluid is likely to be close to zero, corresponding to equal numbers of particles and anti-particles. The entropy S_b of a proper Lagrangian volume element V of the fluid is then

$$S_b = \frac{(\rho_b + p_b)V}{T_b} , \tag{249}$$

which agrees with the earlier expression (231), but now has the correct normalization.

5.27.4 Entropy creation

One fundamentally interesting question about black hole interiors is how much entropy might be created inside the horizon. Bekenstein first argued that a black hole should have a quantum entropy proportional to its horizon area A, and Hawking (1974) supplied the constant of proportionality 1/4 in Planck units. The Bekenstein-Hawking entropy $S_{\rm BH}$ is, in Planck units $c = G = \hbar = 1$,

$$S_{\rm BH} = \frac{A}{4} . \tag{250}$$

For a spherical black hole of horizon radius r_+, the area is $A = 4\pi r_+^2$. Hawking showed that a black hole has a temperature T_{H} equal to $1/(2\pi)$ times the surface gravity g_+ at its horizon, again in Planck units,

$$T_{\mathrm{H}} = \frac{g_+}{2\pi} \ . \tag{251}$$

The surface gravity is defined to be the proper radial acceleration measured by a person in free-fall at the horizon. For a spherical black hole, the surface gravity is $g_+ = -D_t\beta_t = M/r^2 + 4\pi r p$ evaluated at the horizon, equation (161a).

The proper velocity of the baryonic fluid through the sonic point equals ξ^x/ξ^η, equation (187). Thus the entropy S_b accreted through the sonic point per unit proper time of the fluid is

$$\frac{dS_b}{d\tau} = \frac{(1+w_b)\rho_b}{T_b} \frac{4\pi r^2 \xi^x}{\xi^\eta} \ . \tag{252}$$

The horizon radius r_+, which is at fixed conformal radius x, expands in proportion to the conformal factor, $r_+ \propto a$, and the conformal factor a increases as $d\ln a/d\tau = \partial_\eta \ln a = \mathrm{v}/(r\xi^\eta)$, so the Bekenstein-Hawking entropy $S_{\mathrm{BH}} = \pi r_+^2$ increases as

$$\frac{dS_{\mathrm{BH}}}{d\tau} = \frac{2\pi r_+^2 \mathrm{v}}{r\xi^\eta} \ . \tag{253}$$

Thus the entropy S_b accreted through the sonic point per unit increase of the Bekenstein-Hawking entropy S_{BH} is

$$\frac{dS_b}{dS_{\mathrm{BH}}} = \left. \frac{(1+w_b)\rho_b 4\pi r^3 \xi^x}{2\pi r_+^2 \mathrm{v} T_b} \right|_{r=r_s} \ . \tag{254}$$

Inside the sonic point, dissipation increases the entropy according to equation (230). Since the entropy can diverge at a central singularity where the density diverges, and quantum gravity presumably intervenes at some point, it makes sense to truncate the production of entropy at a "splat" point where the density ρ_b hits a prescribed splat density $\rho_\#$

$$\rho_b = \rho_\# \ . \tag{255}$$

Integrating equation (230) from the sonic point to the splat point yields the ratio of the entropies at the sonic and splat points. Multiplying the accreted entropy, equation (254), by this ratio yields the rate of increase of the entropy of the black hole, truncated at the splat point, per unit increase of its Bekenstein-Hawking entropy

$$\frac{dS_b}{dS_{\mathrm{BH}}} = \left. \frac{(1+w_b)\rho_b 4\pi r^3 \xi^x}{2\pi r_+^2 \mathrm{v} T_b} \right|_{\rho_b=\rho_\#} \ . \tag{256}$$

5.27.5 Holography

The idea that the entropy of a black hole cannot exceed its Bekenstein-Hawking entropy has motivated **holographic** conjectures that the degrees of freedom of a volume are somehow encoded on its boundary, and consequently that the entropy of a volume is bounded by those degrees of freedom. Various counter-examples dispose of most simple-minded versions of holographic entropy bounds. The most successful entropy bound, with no known counter-examples, is Bousso's **covariant entropy bound** (Bousso 2002, Rev. Mod. Phys. 74, 825). The covariant entropy bound concerns not just any old 3-dimensional volume, but rather the 3-dimensional volume formed by a null hypersurface, a lightsheet. For example, the horizon of a black hole is a null hypersurface, a lightsheet. The covariant entropy bound asserts that the entropy that passes (inward or outward) through a lightsheet that is everywhere converging cannot exceed 1/4 of the 2-dimensional area of the boundary of the lightsheet.

In the self-similar black holes under consideration, the horizon is expanding, and outgoing lightrays that sit on the horizon do not constitute a converging lightsheet. However, a spherical shell of ingoing lightrays that starts on the horizon falls inwards and therefore does form a converging lightsheet, and a spherical shell of outgoing lightrays that starts just slightly inside the horizon also falls inward and forms a converging lightsheet. The rate at which entropy S_b passes through such ingoing or outgoing spherical lightsheets per unit decrease in the area $S_{\rm cov} \equiv \pi r^2$ of the lightsheet is

$$\left| \frac{dS_b}{dS_{\rm cov}} \right| = \frac{dS_b}{dS_{\rm BH}} \frac{r_+^2}{r^2} \frac{\mathrm{v}}{\xi^x |\beta_\eta \mp \beta_x|} \ , \tag{257}$$

in which the $\mp$ sign is $-$ for ingoing, $+$ for outgoing lightsheets. A sufficient condition for Bousso's covariant entropy bound to be satisfied is

$$|dS_b/dS_{\rm cov}| \leq 1 \ . \tag{258}$$

5.27.6 Black hole accreting a neutral relativistic plasma

The simplest case to consider is that of a black hole accreting a neutral relativistic plasma. In the self-similar solutions, the charge of the black hole is produced self-consistently by the accreted charge of the baryonic fluid, so a neutral fluid produces an uncharged black hole.

Figure 4 shows the baryonic density ρ_b and Weyl curvature C inside the uncharged black hole. The mass and accretion rate have been taken to be

$$M_\bullet = 4 \times 10^6 \, \mathrm{M}_\odot \ , \quad \dot{M}_\bullet = 10^{-16} \ , \tag{259}$$

which are motivated by the fact that the mass of the supermassive black hole at the center of the Milky Way is $4 \times 10^6 \, \mathrm{M}_\odot$, and its accretion rate is

$$\frac{\text{Mass of MW black hole}}{\text{age of Universe}} \approx \frac{4 \times 10^6 \, \mathrm{M}_\odot}{10^{10} \, \mathrm{yr}} \approx \frac{6 \times 10^{60} \text{ Planck units}}{4 \times 10^{44} \text{ Planck units}} \approx 10^{-16} \ . \tag{260}$$

Figure 4 shows that the baryonic plasma plunges uneventfully to a central singularity, just as in the Schwarzschild solution. The Weyl curvature scalar hits the Planck scale, $|C| = 1$, while the baryonic proper density ρ_b is still well below the Planck density, so this singularity is curvature-dominated.

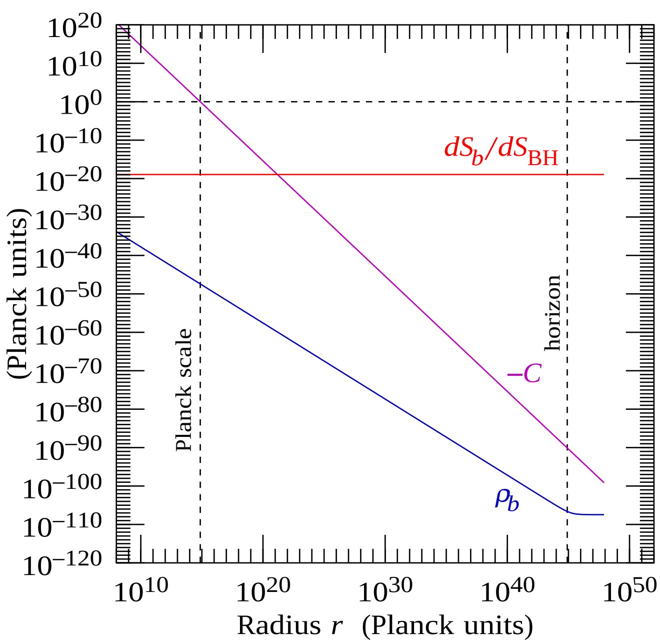

Figure 4: An uncharged baryonic plasma falls into an uncharged spherical black hole. The plot shows in Planck units, as a function of radius, the plasma density ρ_b, the Weyl curvature scalar C (which is negative), and the rate dS_b/dS_{BH} of increase of the plasma entropy per unit increase in the Bekenstein-Hawking entropy of the black hole. The mass is $M_\bullet = 4 \times 10^6 \, \mathrm{M}_\odot$, the accretion rate is $\dot{M}_\bullet = 10^{-16}$, and the equation of state is $w_b = 0.32$.

5.27.7 Black hole accreting a non-conducting charged relativistic plasma

The next simplest case is that of a black hole accreting a charged but non-conducting relativistic plasma.

Figure 5 shows a black hole with charge-to-mass $Q_\bullet / M_\bullet = 10^{-5}$, but otherwise the same parameters as in the uncharged black hole of §5.27.6: $M_\bullet = 4 \times 10^6\,\mathrm{M}_\odot$, $\dot{M}_\bullet = 10^{-16}$, and $w_b = 0.32$. Inside the outer horizon, the baryonic plasma, repelled by the electric charge of the black hole self-consistently generated by the accretion of the charged baryons, becomes outgoing. Like the Reissner-Nordström geometry, the black hole has an (outgoing) inner horizon. The baryons drop through the inner horizon, shortly after which the self-similar solution terminates at an irregular sonic point, where the proper acceleration diverges. Normally this is a signal that a shock must form, but even if a shock is introduced, the plasma still terminates at an irregular sonic point shortly downstream of the shock. The failure of the self-similar to continue does not invalidate the solution, because the failure is hidden beneath the inner horizon, and cannot be communicated to infalling matter above it.

The solution is nevertheless not realistic, because it assumes that there is no ingoing matter, such as would inevitably be produced for example by infalling neutral dark matter. Such ingoing matter would appear infinitely blueshifted to the outgoing baryons falling through the inner horizon, which would produce mass inflation, as in §5.27.10.

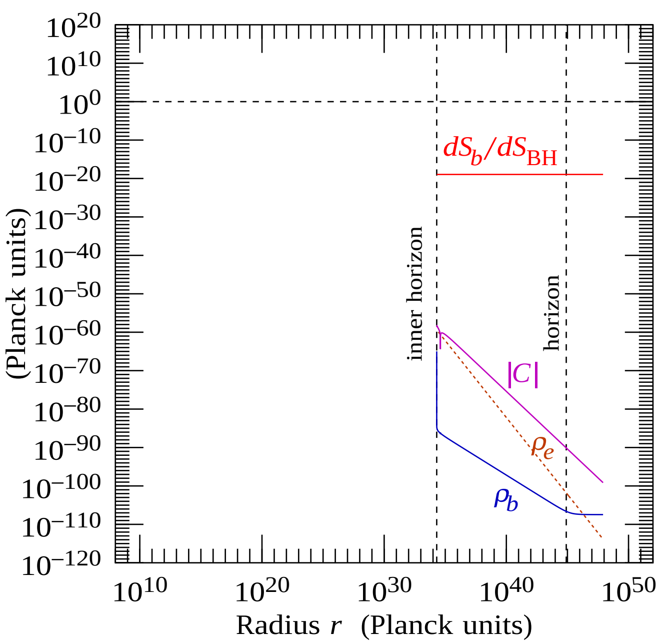

Figure 5: A plasma that is charged but non-conducting. The black hole has an inner horizon like the Reissner-Nordström geometry. The self-similar solution terminates at an irregular sonic point just beneath the inner horizon. The mass is $M_\bullet = 4 \times 10^6\,\mathrm{M}_\odot$, accretion rate $\dot{M}_\bullet = 10^{-16}$, equation of state $w_b = 0.32$, and black hole charge-to-mass $Q_\bullet / M_\bullet = 10^{-5}$.

5.27.8 Black hole accreting a conducting relativistic plasma

What happens if the baryonic plasma is not only electrically charged but also electrically conducting? If the conductivity is small, then the solutions resemble the non-conducting solutions of the previous subsection, §5.27.7. But if the conductivity is large enough effectively to neutralize the plasma as it approaches the center, then the plasma can plunge all the way to the central singularity, as in the uncharged case in §5.27.6.

Figure 6 shows a case in which the conductivity has been tuned to equal, within numerical accuracy, the critical conductivity $\kappa_b = 0.35$ above which the plasma collapses to a central singularity. The parameters are otherwise the same as in previous subsections: a mass of $M_\bullet = 4 \times 10^6\,\mathrm{M}_\odot$, an accretion rate $\dot{M}_\bullet = 10^{-16}$, an equation of state $w_b = 0.32$, and a black hole charge-to-mass of $Q_\bullet/M_\bullet = 10^{-5}$.

The solution at the critical conductivity exhibits the periodic self-similar behavior first discovered in numerical simulations by Choptuik (1993, PRL 70, 9), and known as "critical collapse" because it happens at the borderline between solutions that do and do not collapse to a black hole. The ringing of curves in Figure 6 is a manifestation of the self-similar periodicity, not a numerical error.

These solutions are not subject to the mass inflation instability, and they could therefore

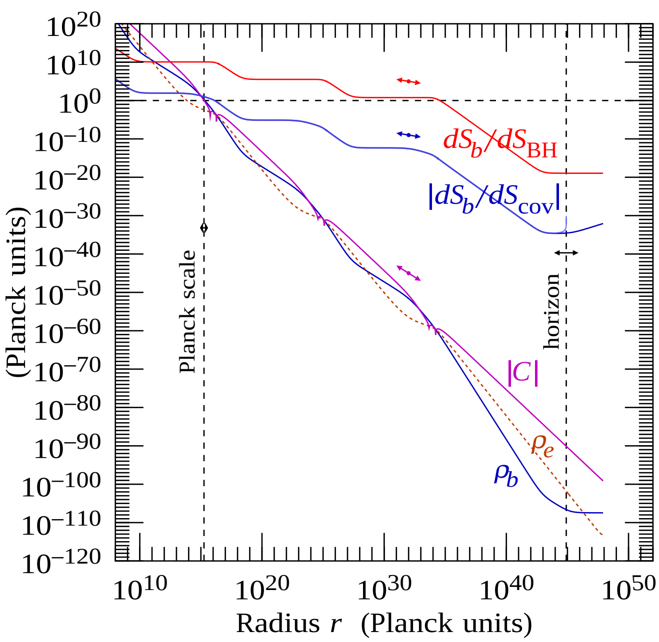

Figure 6: Here the baryonic plasma is charged, and electrically conducting. The conductivity is at (within numerical accuracy) the threshold above which the plasma plunges to a central singularity. The mass is $M_\bullet = 4 \times 10^6\,\mathrm{M}_\odot$, the accretion rate $\dot{M}_\bullet = 10^{-16}$, the equation of state $w_b = 0.32$, the charge-to-mass $Q_\bullet/M_\bullet = 10^{-5}$, and the conductivity parameter $\kappa_b = 0.35$. Arrows show how quantities vary a factor of 10 into the past and future.

be prototypical of the behavior inside realistic rotating black holes. For this to work, the outward transport of angular momentum inside a rotating black hole must be large enough effectively to produce zero angular momentum at the center. My instinct is that angular momentum transport is probably not strong enough, but I do not know this for sure. If angular momentum transport is not strong enough, then mass inflation will take place.

Figure 6 shows that the entropy produced by Ohmic dissipation inside the black hole can potentially exceed the Bekenstein-Hawking entropy of the black hole by a large factor. The Figure shows the rate $dS_b/dS_{\rm BH}$ of increase of entropy per unit increase in its Bekenstein-Hawking entropy, as a function of the hypothetical splat point above which entropy production is truncated. The rate is almost independent of the black hole mass $M_\bullet$ at fixed splat density $\rho_\#$, so it is legitimate to interpret $dS_b/dS_{\rm BH}$ as the cumulative entropy created inside the black hole relative to the Bekenstein-Hawking entropy. Truncated at the Planck scale, $|C| = 1$, the entropy relative to Bekenstein-Hawking is $dS_b/dS_{\rm BH} \approx 10^{10}$.

Generally, the smaller the accretion rate $\dot{M}_\bullet$, the more entropy is produced. If moreover the charge-to-mass $Q_\bullet/M_\bullet$ is large, then the entropy can be produced closer to the outer horizon. Figure 7 shows a model with a relatively large charge-to-mass $Q_\bullet/M_\bullet = 0.8$, and a low accretion rate $\dot{M}_\bullet = 10^{-28}$. The large charge-to-mass ratio in spite of the relatively high conductivity requires force-feeding the black hole: the sonic point must be pushed to just above the horizon. The large charge and high conductivity leads to a burst of entropy

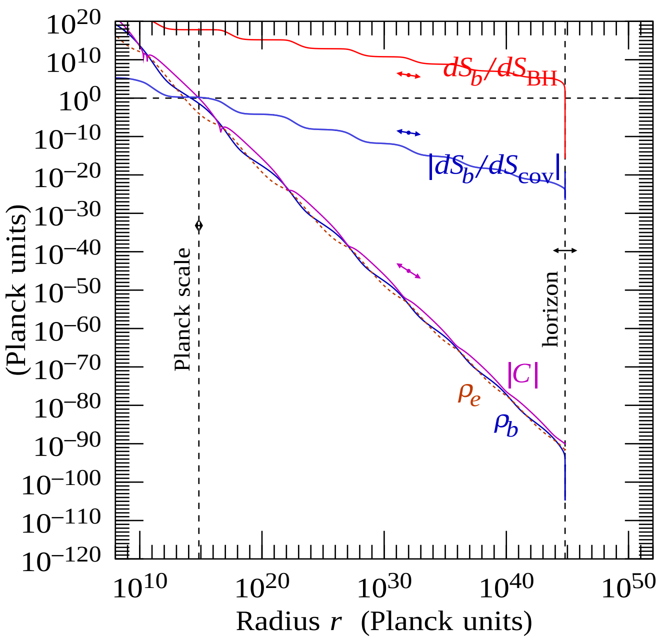

Figure 7: This black hole creates a lot of entropy by having a large charge-to-mass $Q_\bullet/M_\bullet = 0.8$ and a low accretion rate $\dot{M}_\bullet = 10^{-28}$. The conductivity parameter $\kappa_b = 0.35$ is again at the threshold above which the plasma plunges to a central singularity. The equation of state is $w_b = 0.32$.

production just beneath the horizon.

If the entropy created inside a black hole exceeds the Bekenstein-Hawking entropy, and the black hole later evaporates radiating only the Bekenstein-Hawking entropy, then entropy is destroyed, violating the second law of thermodynamics.

This startling conclusion is premised on the assumption that entropy created inside a black hole accumulates additively, which in turn derives from the assumption that the Hilbert space of states is multiplicative over spacelike-separated regions. This assumption, called **locality**, derives from the fundamental proposition of quantum field theory in flat space that field operators at spacelike-separated points commute. This reasoning is essentially the same as originally led Hawking (1976) to conclude that black holes must destroy information.

The same ideas that motivate holography also rescue the second law. If the future lightcones of spacelike-separated points do not intersect, then the points are permanently out of communication, and can behave like alternate quantum realities, like Schrödinger's dead-and-alive quantum cat. Just as it is not legitimate to the add the entropies of the dead cat and the live cat, so also it is not legitimate to add the entropies of regions inside a black hole whose future lightcones do not intersect. The states of such separated regions, instead of being distinct, are quantum entangled with each other.

Figures 6 and 7 show that the rate $|dS_b/dS_{\rm cov}|$ at which entropy passes through ingoing or outgoing spherical lightsheets is less than one at all scales below the Planck scale. This shows not only that the black holes obey Bousso's covariant entropy bound, but also that no individual observer inside the black hole sees more than the Bekenstein-Hawking entropy on their lightcone. No observer actually witnesses a violation of the second law.

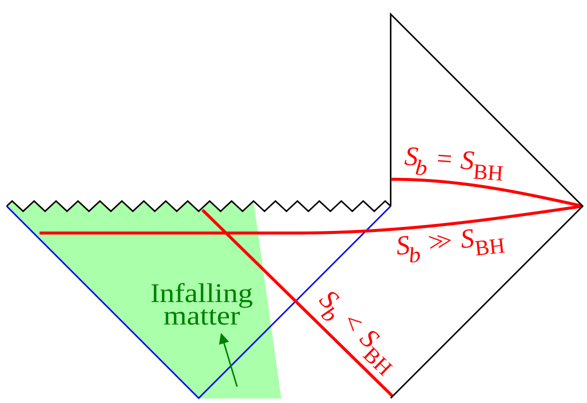

Figure 8: Partial Penrose diagram of the black hole. The entropy passing through the spacelike slice before the black hole evaporates exceeds that passing through the spacelike slice after the black hole evaporates, apparently violating the second law of thermodynamics. However, the entropy passing through any null slice respects the second law.

5.27.9 Black hole accreting a charged massless scalar field

The charged, non-conducting plasma considered in §5.27.7 fell through an (outgoing) inner horizon without undergoing mass inflation. This can be attributed to the fact that relativistic counter-streaming could not occur: there was only a single (outgoing) fluid, and the speed of sound in the fluid was less than the speed of light.

In reality, unless dissipation destroys the inner horizon as in §5.27.8, then relativistic counter-streaming between ingoing and outgoing fluids will undoubtedly take place, through gravitational waves if nothing else.

One way to allow relativistic counter-streaming is to let the speed of sound be the speed of light. This is true in a massless scalar (= spin-0) field ϕ, which has an equation of state $w_\phi = 1$. Figure 9 shows a black hole that accretes a charged, non-conducting fluid with this equation of state. The parameters are otherwise the same as as in previous subsections: a mass of $M_\bullet = 4 \times 10^6\,\mathrm{M}_\odot$, an accretion rate of $\dot{M}_\bullet = 10^{-16}$, and a black hole charge-to-mass of $Q_\bullet/M_\bullet = 10^{-5}$. As the Figure shows, mass inflation takes place just above the place where the inner horizon would be. During mass inflation, the density ρ_ϕ and the Weyl scalar C rapidly exponentiate up to the Planck scale and beyond.

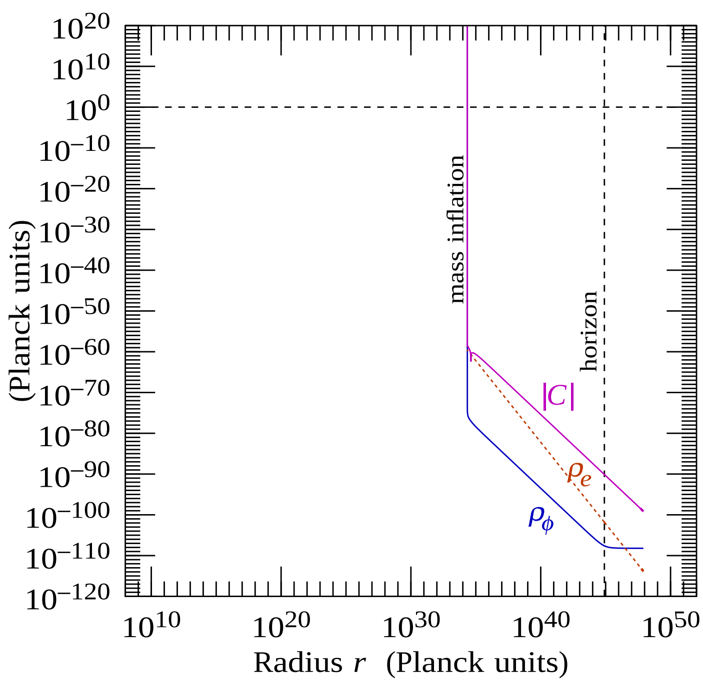

Figure 9: Instead of a relativistic plasma, this shows a charged scalar field ϕ whose equation of state $w_\phi = 1$ means that the speed of sound equals the speed of light. The scalar field therefore supports relativistic counter-streaming, as a result of which mass inflation occurs just above the erstwhile inner horizon. The mass is $M_\bullet = 4 \times 10^6\,\mathrm{M}_\odot$, the accretion rate $\dot{M}_\bullet = 10^{-16}$, the charge-to-mass $Q_\bullet/M_\bullet = 10^{-5}$, and the conductivity is zero.

One of the remarkable features of the mass inflation instability is that the smaller the accretion rate, the more violent the instability. Figure 10 shows mass inflation in a black hole of charge-to-mass $Q_\bullet/M_\bullet = 0.8$ accreting a massless scalar field at rates $\dot{M}_\bullet = 0.01$, 0.003, and 0.001. The charge-to-mass has been chosen to be largish so that the inner horizon is not too far below the outer horizon, and the accretion rates have been chosen to be large because otherwise the inflationary growth rate is too rapid to be discerned easily on the graph. The density ρ_ϕ and Weyl scalar C exponentiate along with, and in proportion to, the interior mass M, which increases as the radius r decreases as

$$M \propto \exp(-\ln r/\dot{M}_\bullet) \ . \tag{261}$$

Physically, the scale of length of inflation is set by how close to the inner horizon infalling material approaches before mass inflation begins. The smaller the accretion rate, the closer the approach, and consequently the shorter the length scale of inflation.

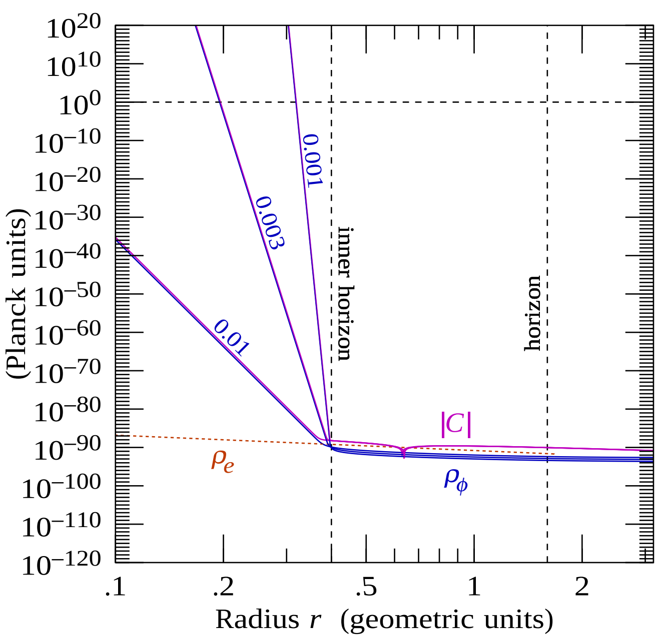

Figure 10: The density ρ_ϕ and Weyl curvature scalar $|C|$ inside a black hole accreting a massless scalar field. The graph shows three cases, with mass accretion rates $\dot{M}_\bullet = 0.01$, 0.003, and 0.001. The graph illustrates that the smaller the accretion rate, the faster the density and curvature inflate. Mass inflation destroys the inner horizon: the dashed vertical line labeled "inner horizon" shows the position that the inner horizon would have if mass inflation did not occur. The black hole mass is $M_\bullet = 4 \times 10^6 \, \mathrm{M}_\odot$, the charge-to-mass is $Q_\bullet/M_\bullet = 0.8$, and the conductivity is zero.

5.27.10 Black hole accreting charged baryons and dark matter

No scalar field (massless or otherwise) has yet been observed in nature, although it is supposed that the Higgs field is a scalar field, and it is likely that cosmological inflation was driven by a scalar field. Another way to allow mass inflation in simple models is to admit not one but two fluids that can counter-stream relativistically through each other. A natural possibility is to feed the black hole not only with a charged relativistic fluid of baryons but also with neutral pressureless dark matter that streams freely through the baryons. The charged baryons, being repelled by the electric charge of the black hole, become outgoing, while the neutral dark matter remains ingoing.

Figure 11 shows that relativistic counter-streaming between the baryons and the dark matter causes the center-of-mass density ρ and the Weyl curvature scalar C to inflate quickly up to the Planck scale and beyond. The ratio of dark matter to baryonic density at the sonic point is $\rho_d/\rho_b = 0.1$, but otherwise the parameters are the generic parameters of previous subsections: $M_\bullet = 4 \times 10^6\,\mathrm{M}_\odot$, $\dot{M}_\bullet = 10^{-16}$, $w_b = 0.32$, $Q_\bullet/M_\bullet = 10^{-5}$, and zero conductivity. Almost all the center-of-mass energy ρ is in the counter-streaming energy between the outgoing baryonic and ingoing dark matter. The individual densities ρ_b of baryons and ρ_d of dark matter (and ρ_e of electromagnetic energy) increase only modestly.

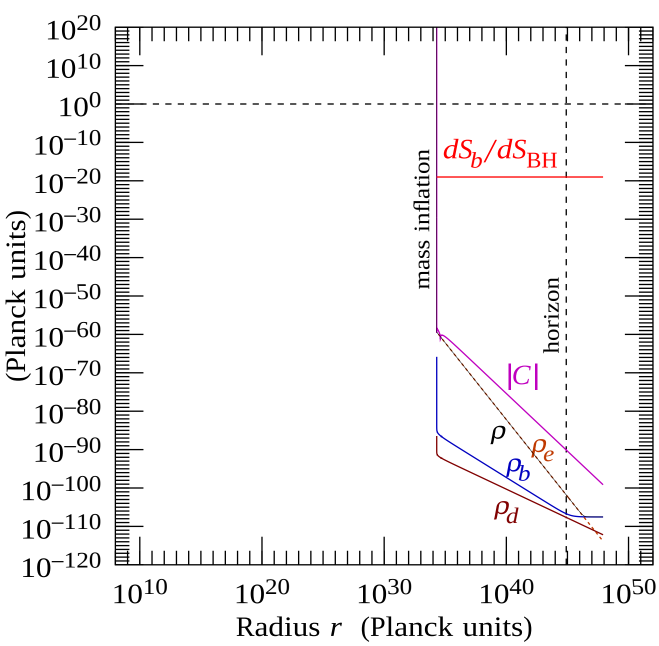

Figure 11: Back to the relativistic charged baryonic plasma, but now in addition the black hole accretes neutral pressureless uncharged dark matter, which streams freely through the baryonic plasma. The relativistic counter-streaming produces mass inflation just above the erstwhile inner horizon. The mass is $M_\bullet = 4 \times 10^6\,\mathrm{M}_\odot$, the accretion rate $\dot{M}_\bullet = 10^{-16}$, the baryonic equation of state $w_b = 0.32$, the charge-to-mass $Q_\bullet/M_\bullet = 10^{-5}$, the conductivity is zero, and the ratio of dark matter to baryonic density at the outer sonic point is $\rho_d/\rho_b = 0.1$.

As in the case of the massless scalar field considered in the previous subsection, §5.27.9, the smaller the accretion rate, the shorter the length scale of inflation. Not only that, but the smaller one of the ingoing or outgoing streams is relative to the other, the shorter the length scale of inflation. Figure 12 shows a black hole with three different ratios of the dark-matter-to-baryon density ratio at the sonic point, $\rho_d/\rho_b = 0.3$, 0.1, and 0.03, all with the same total accretion rate $\dot{M}_\bullet = 10^{-2}$. The smaller the dark matter stream, the faster is inflation. The accretion rate $\dot{M}_\bullet$ and the dark-matter-to-baryon ratio ρ_d/ρ_b have been chosen to be relatively large so that the inflationary growth rate is discernable easily on the graph.

Figure 12 shows that, as in Figure 11, almost all the center-of-mass energy is in the streaming energy between the baryons and the dark matter. For one case, $\rho_d/\rho_b = 0.3$, Figure 12 shows the individual densities ρ_b of baryons, ρ_d of dark matter, and ρ_e of electromagnetic energy, all of which remain tiny compared to the streaming energy.

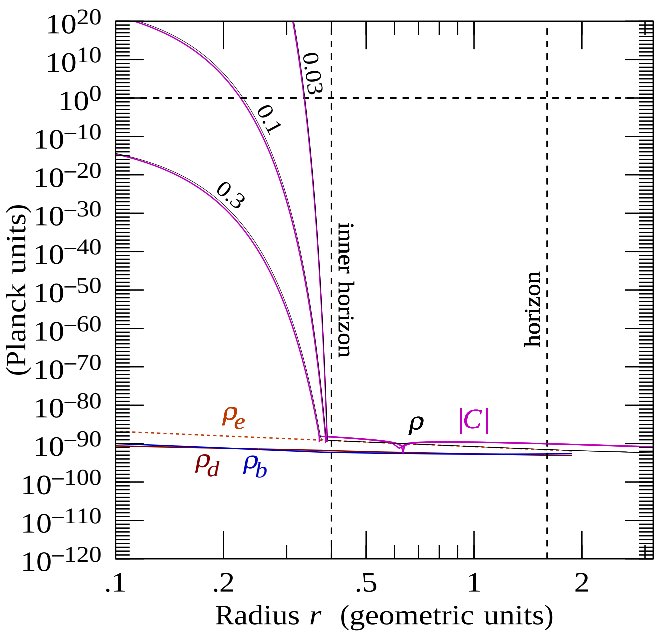

Figure 12: The center-of-mass density ρ and Weyl curvature $|C|$ inside a black hole accreting baryons and dark matter at rate $\dot{M}_\bullet = 0.01$. The graph shows three cases, with dark-matter-to-baryon ratio at the sonic point of $\rho_d/\rho_b = 0.3$, 0.1, and 0.03. The smaller the ratio of dark matter to baryons, the faster the center-of-mass density ρ and curvature C inflate. For the largest ratio, $\rho_d/\rho_b = 0.3$ (to avoid confusion, only this case is plotted), the graph also shows the individual proper densities ρ_b of baryons, ρ_d of dark matter, and ρ_e of electromagnetic energy. During mass inflation, almost all the center-of-mass energy ρ is in the streaming energy: the proper densities of individual components remain small. The black hole mass is $M_\bullet = 4 \times 10^6\,\mathrm{M}_\odot$, the baryonic equation of state is $w_b = 0.32$, the charge-to-mass is $Q_\bullet/M_\bullet = 0.8$, and the conductivity is zero.

5.27.11 The black hole particle accelerator

The previous subsection, §5.27.10, showed that almost all the center-of-mass energy during mass inflation is in the energy of counter-streaming. Thus the black hole acts like an extravagantly powerful particle accelerator.

Mass inflation is an exponential instability. The nature of the black hole particle accelerator is that an individual particle spends approximately an equal interval of proper time being accelerated through each decade of collision energy.

Each baryon in the black hole collider sees a flux $n_d u^r$ of dark matter particles per unit area per unit time, where $n_d = \rho_d/m_d$ is the proper number density of dark matter particles in their own frame, and u^r is the radial component of the proper 4-velocity, the γv, of the dark matter through the baryons. The γ factor in u^r is the relavistic beaming factor: all frequencies, including the collision frequency, are speeded up by the relativistic beaming factor γ. As the baryons accelerate through the collider, they spend a proper time interval

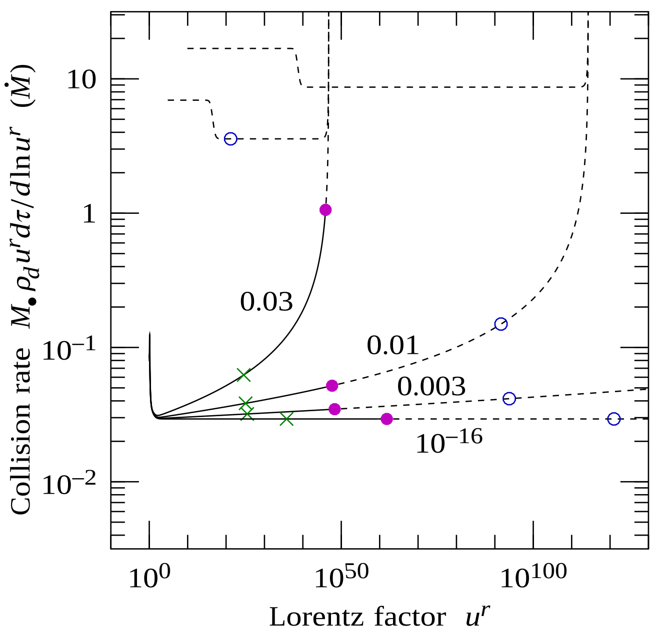

Figure 13: Collision rate of the black hole collider per e-fold of Lorentz factor, for various mass accretion rates: $\dot{M}_\bullet = 0.03$, 0.01, 0.003, and 10^{-16}. The dark-matter-to-baryon ratio at the sonic point is $\rho_d/\rho_b = 0.1$ in all cases, and the black hole charge-to-mass is $Q_\bullet/M_\bullet = 0.8$, The various symbols show where different quantum effects, all of which are neglected in this calculation, are expected to come into play: crosses show where the Unruh temperature exceeds the plasma temperature; filled circles show where the Weyl curvature scalar C exceeds the Plank scale; open circles show where the Unruh temperature exceeds the Planck temperature. The black hole mass is $M_\bullet = 4 \times 10^6\,\mathrm{M}_\odot$, the equation of state is $w_b = 0.32$, and the conductivity is zero.

$d\tau/d\ln u^r$ in each e-fold of Lorentz factor u^r. The number of collisions per baryon per e-fold of u^r is the dark matter flux $(\rho_d/m_d)u^r$, multiplied by the time $d\tau/d\ln u^r$, multiplied by the collision cross-section σ. The total cumulative number of collisions that have happened in the black hole particle collider equals this multiplied by the total number of baryons that have fallen into the black hole, which is approximately equal to the black hole mass $M_\bullet$ divided by the mass m_b per baryon. Thus the total cumulative number of collisions in the black hole collider is

$$\frac{\text{number of collisions}}{e\text{-fold of } u^r} = \frac{M_\bullet}{m_b}\frac{\rho_d}{m_d}\sigma u^r \frac{d\tau}{d\ln u^r} \ . \tag{262}$$

Figure 13 shows, for several different accretion rates $\dot{M}_\bullet$, the collision rate $M_\bullet \rho_d u^r d\tau/d\ln u^r$ of the black hole collider, expressed in units of the black hole accretion rate $\dot{M}_\bullet$. This collision rate, multiplied by $\dot{M}_\bullet \sigma/(m_d m_b)$, gives the number of collisions (262) in the black hole. In the units $c = G = 1$ being used here, the mass of a baryon (proton) is $1\,\text{GeV} \approx 10^{-54}\,\text{m}$. If the cross-section σ is expressed in canonical accelerator units of femtobarns ($1\,\text{fb} = 10^{-43}\,\text{m}^2$) then the number of collisions (262) is

$$\frac{\text{number of collisions}}{e\text{-fold of } u^r} = 10^{45}\left(\frac{\sigma}{1\,\text{fb}}\right)\left(\frac{300\,\text{GeV}^2}{m_b m_d}\right)\left(\frac{\dot{M}_\bullet}{10^{-16}}\right)\left(\frac{M_\bullet \rho_d u^r d\tau/d\ln u^r}{0.03\,\dot{M}_\bullet}\right) \ . \tag{263}$$

Epoch 2008 particle accelerators are proud of having delivered an inverse femtobarn of luminosity. Equation (263) shows that the black hole accelerator delivers about 10^{45} more collisions than that, and it does so in each e-fold of collision energy up to the Planck energy and beyond.

The calculations illustrated in Figure 13 ignore any quantum effects. The gravitational acceleration induces a pressure gradient in the baryons, which causes the baryons to experience a proper acceleration in their own frame. As first shown by Unruh (1976), a frame undergoing proper acceleration g will perceive itself to be engulfed in black body radiation with Unruh temperature

$$T_{\text{U}} = \frac{g}{2\pi} \tag{264}$$

in Planck units. Figure 13 shows where the Unruh temperature T_{U} exceeds first the baryon temperature T_b, and subsequently the Planck temperature T_{p}. In between these two events, the Weyl curvature scalar C exceeds the Planck scale. Above this scale quantum gravity presumably comes into play, invalidating the calculation, and the lines in Figure 13 in this region are therefore shown dashed.

5.28 ADM formalism

The Arnowitt-Deser-Misner (1962) formalism, also known as the 3+1 formalism, is widely used in numerical general relativity. For a review, see L. Lehner (2001, CQG 18, R25).

5.28.1 ADM tetrad

The ADM formalism splits the spacetime coordinates x^μ into a time coordinate t and spatial coordinates x^i, $i = 1, 2, 3$,

$$x^\mu \equiv \{t, x^i\} \ , \tag{265}$$

and evolves the spacetime from one hypersurface of constant time, $t =$ constant, to the next. At each point of spacetime, the hypersurface of constant time has a unique unit normal $\boldsymbol{\gamma}_t$, defined to have unit length and to be orthogonal to the spatial tangent axes $\boldsymbol{g}_i$

$$\boldsymbol{\gamma}_t \cdot \boldsymbol{\gamma}_t = 1 \ , \quad \boldsymbol{\gamma}_t \cdot \boldsymbol{g}_i = 0 \quad i = 1, 2, 3 \ . \tag{266}$$

The ADM approach is to work in a tetrad frame $\boldsymbol{\gamma}_m$ consisting of the time axis $\boldsymbol{\gamma}_t$ together with the three original spatial tangent axes $\boldsymbol{g}_i$

$$\boldsymbol{\gamma}_m \equiv \{\boldsymbol{\gamma}_t, \boldsymbol{\gamma}_i\} \equiv \{\boldsymbol{\gamma}_t, \boldsymbol{g}_i\} \ . \tag{267}$$

The tetrad metric γ_{mn} in the ADM formalism is thus

$$\gamma_{mn} = \begin{pmatrix} 1 & 0 \\ 0 & \gamma_{ij} \end{pmatrix} \ , \tag{268}$$

whose time part is orthonormal, and whose spatial part is the same as the spatial part of the coordinate metric, $\gamma_{ij} = g_{ij}$ for $i, j = 1, 2, 3$. The inverse tetrad metric γ^{mn} is correspondingly

$$\gamma^{mn} = \begin{pmatrix} 1 & 0 \\ 0 & \gamma^{ij} \end{pmatrix} \tag{269}$$

whose spatial part γ^{ij} is the inverse of γ_{ij}, but is *not* the same as the spatial part g^{ij} of the inverse coordinate metric $g^{\mu\nu}$. One might say that the ADM approach is semi-tetrad (my word). The time axis $\boldsymbol{\gamma}_t$ is unchanged by a spatial coordinate transformation $x^i \to x'^i$ (one that leaves the time coordinate t unchanged), and is therefore a spatial coordinate scalar. The time axis $\boldsymbol{\gamma}_t$ is changed by a temporal coordinate transformation $t \to t'$, and since by definition $\boldsymbol{\gamma}_t$ remains of unit length, that transformation must be a Lorentz transformation. The spatial axes $\boldsymbol{\gamma}_i \equiv \boldsymbol{g}_i$ are changed by both spatial and temporal coordinate transformations.

Because the definition of the time axis $\boldsymbol{\gamma}_t$ is tied to the coordinates, the ADM formalism does not have a concept of tetrad transformations distinct from coordinate transformations. Nevertheless, the usual notion of tensor applies: a quantity is a tetrad tensor if and only if its components transform under (spatial and temporal) coordinate transformations like (products of) the tetrad axes $\boldsymbol{\gamma}_m \equiv \{\boldsymbol{\gamma}_t, \boldsymbol{g}_i\}$. In the ADM formalism there are also spatial coordinate tensors that are not full tetrad tensors. A quantity is a spatial coordinate tensor if and only if it transforms under purely spatial coordinate transformations like (products of) the coordinate spatial axes $\boldsymbol{\gamma}_i \equiv \boldsymbol{g}_i$. In the ADM formalism, a tetrad tensor is necessarily also a spatial coordinate tensor (why?).

The time axis $\boldsymbol{\gamma}_t$ must be some linear combination of the tangent axes $\boldsymbol{g}_\mu$

$$\boldsymbol{\gamma}_t \equiv \frac{1}{\alpha} \left(\boldsymbol{g}_t + \beta^i \boldsymbol{g}_i \right) \ . \tag{270}$$

The quantity α is called the **lapse**, while β^i is called the **shift**. The lapse α is a spatial coordinate scalar, and the shift β^i is a spatial coordinate 3-vector, but neither the shift nor the lapse is a full tetrad-frame tensor. The vierbein $e_m{}^\mu$ and inverse vierbein $e^m{}_\mu$ are

$$e_m{}^\mu = \frac{1}{\alpha} \begin{pmatrix} 1 & \beta^1 & \beta^2 & \beta^3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ , \quad e^m{}_\mu = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ -\beta^1 & 1 & 0 & 0 \\ -\beta^2 & 0 & 1 & 0 \\ -\beta^3 & 0 & 0 & 1 \end{pmatrix} \ . \tag{271}$$

In the usual tetrad approach, the vierbein row $e_t{}^\mu = (1/\alpha)\{1, \beta^i\}$ and the inverse vierbein column $e^m{}_t = \{\alpha, -\beta^i\}$ would be respectively coordinate and tetrad 4-vectors, but this is not true in the ADM formalism because tetrad and coordinate transformations are tied together rather than being independent.

The coordinate metric $ds^2 \equiv \gamma_{mn} e^m{}_\mu e^n{}_\nu dx^\mu dx^\nu$ is

$$ds^2 = \alpha^2 dt^2 + \gamma_{ij} (dx^i - \beta^i dt)(dx^j - \beta^j dt) \ . \tag{272}$$

The coordinate metric $g_{\mu\nu}$ is a coordinate tensor as usual, and in particular it is a spatial coordinate tensor, but it is not a tetrad-frame tensor.

5.28.2 ADM extrinsic curvature

It is customary to introduce the **extrinsic curvature** K_{ij}, a spatial coordinate (but not tetrad) tensor, defined by

$$K_{ij} \equiv \boldsymbol{g}_i \cdot \frac{\partial \boldsymbol{\gamma}_t}{\partial x^j} = \Gamma_{itj} \quad \textit{is} \text{ a spatial coordinate (but not tetrad) tensor} \ , \tag{273}$$

which describes how the unit normal $\boldsymbol{\gamma}_t$ to the 3-dimensional spatial hypersurface changes over the hypersurface, and can therefore be regarded as embodying the curvature of the 3-dimensional spatial hypersurface embedded in the 4-dimensional spacetime. The extrinsic curvature K_{ij} transforms under temporal transformations, but it is neither a coordinate nor tetrad frame tensor under such transformations. The extrinsic curvature is symmetric, $K_{ij} = K_{ji}$, and, from equation (40) with vanishing torsion, has the explicit expression in terms of the directed time derivative $\partial_t \gamma_{ij}$ of the spatial metric, and of the vierbein derivatives d_{lmn}, equation (21),

$$K_{ij} \equiv \tfrac{1}{2} \left(\partial_t \gamma_{ij} + d_{itj} + d_{jti} \right) \ . \tag{274}$$

5.28.3 ADM connections

The non-vanishing tetrad connections are

$$\Gamma_{tit} = -\Gamma_{itt} = -d_{tti} \ , \tag{275a}$$

$$\Gamma_{ijt} = K_{ij} - d_{itj} \ , \tag{275b}$$

$$\Gamma_{itj} = -\Gamma_{tij} = K_{ij} \ , \tag{275c}$$

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial\gamma_{ij}}{\partial x^k} + \frac{\partial\gamma_{ik}}{\partial x^j} - \frac{\partial\gamma_{jk}}{\partial x^i}\right) \ , \tag{275d}$$

where the relevant vierbein derivatives are

$$d_{tti} = -\frac{1}{\alpha}\frac{\partial\alpha}{\partial x^i} \ , \quad d_{itj} = \frac{\gamma_{ik}}{\alpha}\frac{\partial\beta^k}{\partial x^j} \quad \textit{are} \text{ spatial coordinate (not tetrad) tensors} \ . \tag{276}$$

The tetrad connections (275a)–(275c) involving at least one time index t are spatial coordinate tensors, but they are not tetrad tensors. The purely spatial tetrad connections Γ_{ijk}, equation (275d) constitute neither a spatial coordinate tensor nor a tetrad tensor (note that Γ_{ijk}, like the spatial tangent axes $\boldsymbol{g}_i$, transform under temporal coordinate transformations despite the absence of temporal indices).

5.28.4 ADM Riemann tensor

The Riemann tensor is a tetrad tensor. Its components R_{klmn} are

$$R_{titj} = -D_t K_{ij} - K_i^k K_{jk} - \frac{1}{\alpha} D_i^{(3)} D_j^{(3)} \alpha \ , \tag{277a}$$

$$R_{tijk} = D_k^{(3)} K_{ij} - D_j^{(3)} K_{ik} \ , \tag{277b}$$

$$R_{ijkl} = K_{il} K_{jk} - K_{ik} K_{jl} + R_{ijkl}^{(3)} \ , \tag{277c}$$

where the superscript $^{(3)}$ denotes purely spatial, 3D quantities, so that $D_i^{(3)}$ is the covariant spatial derivative, and $R_{ijkl}^{(3)}$ is the spatial Riemann tensor, both considered confined to the 3D spatial hypersurface. The notation D_t for the tetrad-frame covariant time derivative in equation (277a) is a bit of an abuse of notation, because K_{ij} is not a tetrad-frame tensor; $D_t K_{ij}$ signifies what the covariant time derivative of K_{ij} would be if K_{ij} were a tetrad tensor. Specifically,

$$D_t K_{ij} = \partial_t K_{ij} - \Gamma^k_{it} K_{jk} - \Gamma^k_{jt} K_{ik} = \partial_t K_{ij} - 2\, K_i^k K_{jk} + \frac{K_{ik}}{\alpha}\frac{\partial\beta^k}{\partial x^j} + \frac{K_{jk}}{\alpha}\frac{\partial\beta^k}{\partial x^i} \ . \tag{278}$$

The expressions (274) for the extrinsic curvature K_{ij} and (277a) for the Riemann components R_{titj} can also be written

$$K_{ij} \equiv \frac{1}{2\alpha}\left(\frac{\partial\gamma_{ij}}{\partial t} - \mathcal{L}_\beta \gamma_{ij}\right) \ , \tag{279}$$

$$R_{titj} = -\frac{1}{\alpha}\left(\frac{\partial K_{ij}}{\partial t} - \mathcal{L}_\beta K_{ij}\right) + K_i^k K_{jk} - \frac{1}{\alpha} D_i^{(3)} D_j^{(3)} \alpha \ , \tag{280}$$

where $\mathcal{L}_\beta$ is the Lie derivative in the β direction, a concept to be met later in the context of perturbation theory.

5.28.5 ADM Ricci and Einstein tensors

The Ricci tensor $R_{km} \equiv \gamma^{ln} R_{klmn}$ is, like the Riemann tensor, a tetrad tensor. Its components are

$$\begin{aligned} R_{tt} &= -\,\partial_t K - K^{ij} K_{ij} - \frac{\gamma^{ij}}{\alpha} D^{(3)}_i D^{(3)}_j \alpha \ , & (281a) \\ R_{ti} &= D^{(3)}_j K^j_i - D^{(3)}_i K \ , & (281b) \\ R_{ij} &= -\,D_t K_{ij} - K K_{ij} - \frac{1}{\alpha} D^{(3)}_i D^{(3)}_j \alpha + R^{(3)}_{ij} \ , & (281c) \end{aligned}$$

where $K \equiv \gamma^{ij} K_{ij}$, and $R^{(3)}_{ik} \equiv \gamma^{jl} R^{(3)}_{ijkl}$ is the Ricci tensor confined to the 3D spatial hypersurface. The Ricci scalar $R \equiv \gamma^{km} R_{km}$ is

$$R = -\,2\,\partial_t K - K^2 - K^{ij} K_{ij} - \frac{2}{\alpha}\,\gamma^{ij} D^{(3)}_i D^{(3)}_j \alpha + R^{(3)} \ , \qquad (282)$$

where $R^{(3)} \equiv \gamma^{ij} R^{(3)}_{ij}$ is the Ricci scalar confined to the 3D spatial hypersurface.

The Einstein tensor $G_{km} \equiv R_{km} - \frac{1}{2}\gamma_{km} R$ is

$$\begin{aligned} G_{tt} &= \frac{1}{2}\left(K^2 - K_{ij} K^{ij} - R^{(3)}\right) \ , & (283a) \\ G_{ti} &= D^{(3)}_j K^j_i - D^{(3)}_i K \ , & (283b) \\ G_{ij} &= -\,D_t K_{ij} + \gamma_{ij}\,\partial_t K - K K_{ij} + \frac{1}{2}\,\gamma_{ij}\left(K^2 + K^{kl} K_{kl}\right) & \\ &\quad - \frac{1}{\alpha}\left(D^{(3)}_i D^{(3)}_j \alpha - \gamma_{ij}\,\gamma^{kl} D^{(3)}_k D^{(3)}_l \alpha\right) + G^{(3)}_{ij} \ , & (283c) \end{aligned}$$

where $G^{(3)}_{ij} \equiv R^{(3)}_{ij} - \frac{1}{2}\gamma_{ij} R^{(3)}$ is the Einstein tensor confined to the 3D spatial hypersurface.

5.28.6 Integrating the ADM equations

The ADM formalism takes the fundamental spacetime variables to be the 6 components γ_{ij} of the spatial metric tensor. The lapse α and shift β^i are regarded as adjustable quantities, reflecting the 4 coordinate freedoms of general relativity. A variety of approaches have been developed to choose the lapse and shift so as to try to maintain numerical stability.

According to the usual rules for integrating partial differential equations from initial conditions, it is necessary first to set up initial conditions on the spatial hypersurface at initial time $t = 0$ (say). Then the equations can be integrated forward in time. The equations involving time derivatives are: first, equation (274), which gives $\partial\gamma_{ij}/\partial t$ in terms of the extrinsic curvature K_{ij} and other quantities depending on the lapse and shift; and second, equation (283c), which gives $\partial K_{ij}/\partial t$ in terms of the spatial components G_{ij} of the Einstein tensor and a bunch of other spatial quantities. Thus the evolution equations involve second order time derivatives of γ_{ij}.

Besides the evolution equations (and whatever equations one chooses to govern the lapse and shift), there are 4 equations, called the constraint equations, that involve no time derivatives, namely the Einstein equations (283a) for G_{tt} and (283b) for G_{ti}. The initial conditions on the $t = 0$ hypersurface must be arranged to satisfy these 4 constraints, but thereafter the Bianchi identities ensure that the conditions are automatically satisfied, modulo numerical instabilities. Equation (283a) for G_{tt} is called the **Hamiltonian constraint** (or scalar constraint), while equations (283b) for G_{ti} are called the **momentum constraints** (or vector constraints). Setting up the initial conditions to satisfy these constraints is not an easy matter.