

Electronics Lecture 8 – AC circuit analysis using phasors

8.1 Introduction

The previous lecture discussed the transient response of an RC circuit to a step voltage by switching in a battery. This lecture will investigate the frequency response of the same circuit to an AC voltage. Our knowledge of the impedance of a capacitor makes it relatively easy to analyse this circuit by treating it as a voltage divider. The results are best plotted on a logarithmic scale and the dB units we introduced in lecture 3 will prove useful. The results will show that in the frequency domain the RC circuit can behave as a low or high-pass filter (depending on configuration) which rejects high or low frequency input signals. Practical uses of such circuits will be discussed, in particular the transmission of telephone and radio signals a little later. In the following two lectures the method of circuit analysis using complex quantities presented here will be extended to include inductors and eventually to resonant LCR circuits.

8.2 A generalised rule for impedances

The concept of impedance was introduced in lecture 7. Impedance has the units of Ω (can you see why?) but for

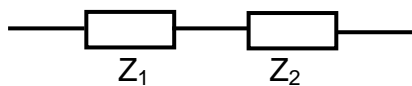


Figure 8.1

capacitors and inductors there is a phase difference between the voltage and current at the terminals of these components, unlike resistors where these are always in phase. Figure 8.1 shows two impedances in series. These may be resistors, capacitors or inductors. We assume for the moment that the total impedance is given by: $Z_{\text{total}} = Z_1 + Z_2$. This is

clearly correct for two resistors as we showed in lecture 2 but what about capacitors? Substituting $Z_1 = 1/j\omega C_1$ and $Z_2 = 1/j\omega C_2$ gives $Z_{\text{total}} = 1/j\omega C_{\text{total}} = 1/j\omega C_1 + 1/j\omega C_2$; removing the constant $j\omega$ term leads directly to the rule for capacitors in series shown in the last lecture. Extending this to the general form for impedances in parallel $1/Z_{\text{total}} = 1/Z_1 + 1/Z_2$ gives the correct result for resistors, and for capacitors gives $C_{\text{total}} = C_1 + C_2$. We could extend this rule to cover combinations of resistors, capacitors (and later) inductors. The results will, in general, yield complex numbers but we already know how to rationalise these to obtain values of the modulus and argument which are real and measurable.

8.3 Analysis of an RC circuit

The circuit shown in figure 8.2 is known as a low pass filter. We will analyse this circuit in the frequency domain, that is, determine its behaviour over a wide range of frequency. Provided the frequency of the voltage

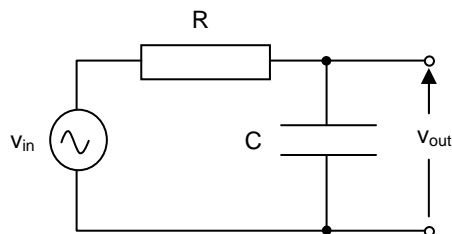


Figure 8.2

source is not too high (say in the GHz range) the resistor will behave as an ideal linear component with a constant value of R . Section 7.6 of the previous lecture showed that the impedance of the capacitor $Z_C = 1/j\omega C$ and the presence of ω in the denominator tells us that the capacitor impedance is low at high frequency and large at low frequency. Consequently we anticipate that for low frequencies where $Z_C \gg R$ most of the input voltage will be dropped across the capacitor whilst at high frequencies $Z_C \ll R$

only a small fraction of v_{in} will be dropped across the capacitor. Therefore we analyse the circuit by monitoring the voltage across the capacitor, labelled v_{out} in the diagram. Note that we have not said anything yet about phase

changes between v_{in} and v_{out} . We will find that the analysis using the complex impedance of the capacitor leads naturally to the changes of phase. Applying KVL

$$v_{in} - v_R - v_C = 0$$

We saw in lecture 2 how this led to the voltage divider rule when considering two resistors. Here, we use the same method but with two impedances to get:

$$v_{out} = \frac{Z_C}{R + Z_C} v_{in} \quad (8.1)$$

We will analyse many circuits like this in the remaining lectures and it is common to use the ratio v_{out}/v_{in} . This is also known as the **voltage transfer ratio** or more generally, the **gain**, **A**. It may seem odd to use the term “gain” since we know that v_{out} will always be smaller than v_{in} making the gain less than unity but you will see when we discuss amplifiers later in the course that the v_{out}/v_{in} ratio can be used quite generally. Note also that **A** is a phasor operator which acts on the phasors representing v_{out} and v_{in} , i.e. $\mathbf{V}_{out} = \mathbf{A}\mathbf{V}_{in}$.

Substituting for Z_C :

$$\frac{v_{out}}{v_{in}} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega CR} \quad (8.2)$$

Where the presence of the j term indicates a phase change between v_{out} and v_{in} . We can rationalise (8.2) by multiplying the numerator and denominator with the complex conjugate of the denominator ($1 - j\omega CR$) to give:

$$\frac{v_{out}}{v_{in}} = \frac{1}{1 + \omega^2 C^2 R^2} - j \frac{\omega CR}{1 + \omega^2 C^2 R^2} \quad (8.3)$$

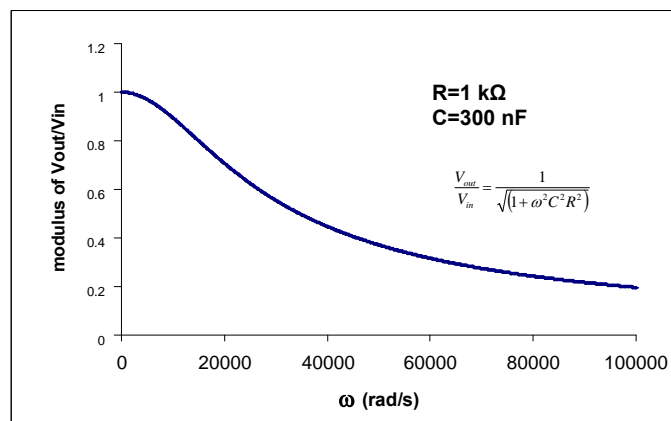
which gives the rectangular ($a + jb$) form of **A**. If we were performing an experiment in the laboratory we would measure v_{out} and v_{in} (or their ratio) and any phase change between them using an oscilloscope. In doing so we would be measuring the modulus and angle of (8.3). These are given by:

$$\left| \frac{v_{out}}{v_{in}} \right| = (a^2 + b^2)^{1/2} = \frac{1}{(1 + \omega^2 C^2 R^2)^{1/2}} \quad \text{and} \quad \tan \phi = b/a = -\omega CR \quad (8.4)$$

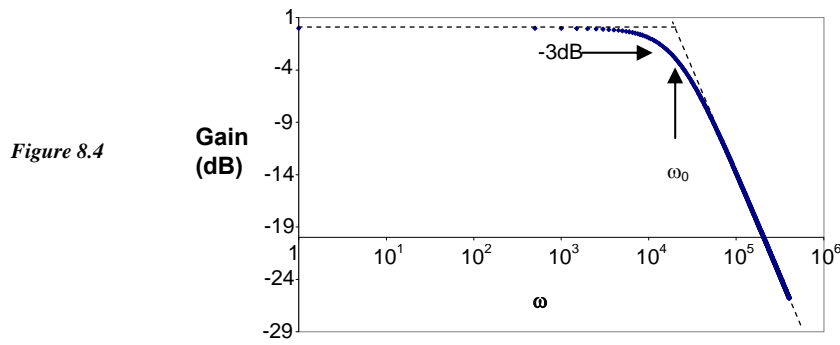
8.4 Frequency dependence of **A** and ϕ - Bode plots

The modulus of v_{out}/v_{in} or gain is plotted in figure 8.3 over a wide frequency range for values $R=1 \text{ k}\Omega$ and $C=300 \text{ nF}$. As we predicted, the gain is close to unity for low frequencies dropping to about 0.2 by 10^5 rad/s . This circuit is known, for obvious reasons, as a *low pass filter* and is used in many applications to limit the bandwidth of signals as will be discussed later. Nevertheless the form of the graph is rather uninteresting. If instead we plot the data on logarithmic axes as shown in figure 8.4 where the ordinate is given in dB units the

Figure 8.3



form of the plot provides us at a glance with the important aspects of the circuit, namely that the gain is 1 (or 0



dB) up to a value slightly greater than 10^4 rad/s above which it drops rapidly. Following on from our time domain analysis of a capacitor we introduced the time constant $\tau = CR$; we do something similar here and assign

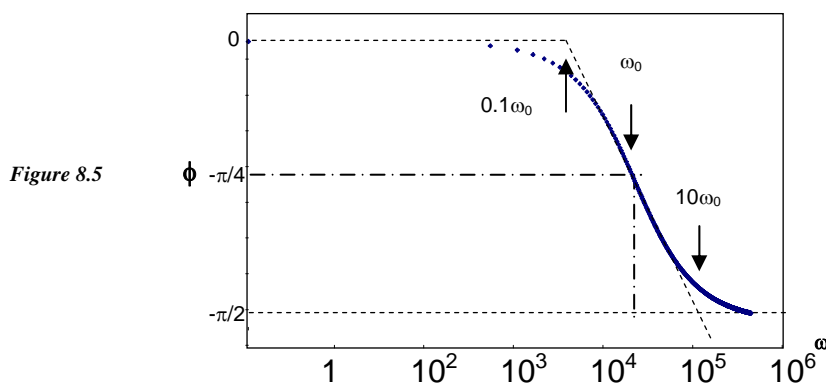
$$\omega_0 = 1/CR. \text{ At } \omega = \omega_0 \text{ (8.4) becomes } \left| \frac{v_{out}}{v_{in}} \right| = \frac{1}{\left(1 + \frac{\omega^2}{\omega_0^2}\right)^{1/2}} = \frac{1}{\sqrt{2}} = 0.7071 \quad (8.5)$$

Recalling from section 3.4 that $G(dB) = 20 \log_{10} \frac{v_{out}}{v_{in}}$ we see that the gain = - 3dB at ω_0 as indicated on the

graph. The graph can be approximately fitted by two asymptotes which meet at ω_0 . For this reason ω_0 is also called the - 3dB point or the “corner frequency”. Below ω_0 the gain is unity whilst above it decreases with a constant slope. Since this “roll off” occurs at moderately high frequencies where $\omega > \omega_0$ we can approximate (8.5)

by $\left| \frac{v_{out}}{v_{in}} \right| = \text{gain} = \frac{\omega_0}{\omega}$ so for every doubling of the frequency the gain reduces by 2. A reduction by a factor of

2 means a reduction of 6dB and the slope has a value of -6dB per octave. Similar reasoning gives the value - 20db per decade.



Equation (8.4) shows that the phase change between v_{out} and v_{in} is also dependent on frequency and figure 8.5 shows a plot of ϕ against $\log \omega$. Summarising the three regions of the graph:

- Up to $\sim 0.1\omega_0$ v_{out} and v_{in} are in phase
- above $\sim 10\omega_0$ v_{out} lags v_{in} by $\pi/2$
- for intermediate values there is an almost linear shift and at $\omega = \omega_0$, v_{out} lags v_{in} by $\pi/4$

Amplitude and phase graphs are known collectively as **Bode plots**. Their great advantage is that only the value of ω_0 need be calculated and the form of the plot is easy to draw.

Some further insight can be obtained by looking back at figure 8.2 and applying KVL: $v_{in} = v_R + v_C$. Since the components form a series circuit the instantaneous current through R and flowing into or out of the capacitor terminals must be the same. Let this current be $i = I \cos \omega t$, then $v_R = IR \cos \omega t$. For the capacitor we know that $i = Cdv_C/dt$ so $v_C = 1/C \int I dt$ which is $1/\omega C \sin \omega t$. These functions are shown in figure 8.6 for the case of small

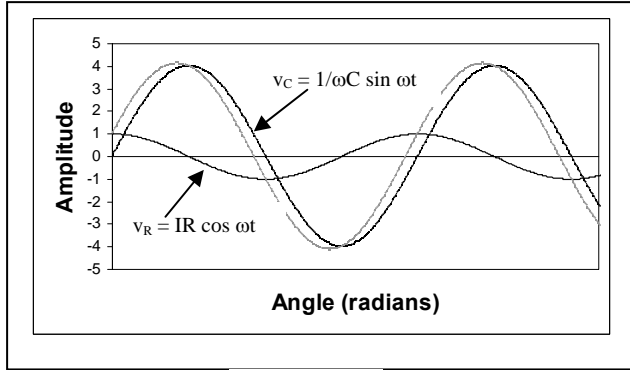
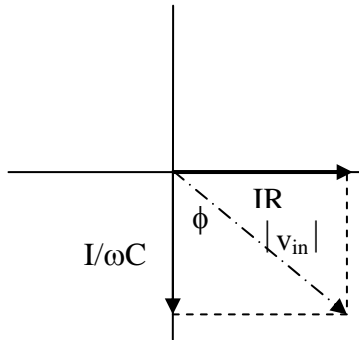


Figure 8.6

ω . Under this condition $Z_C \gg R$ and most of the input voltage is dropped across the capacitor. The input voltage v_{in} is the addition of these two functions and is indicated by the lighter, grey line. We can see immediately that the amplitude of $v_{in} \sim v_C$ as expected and the phase difference is small. It is easy to envisage the situation when ω is large; then $Z_C \ll R$ and the amplitude of $v_{in} \sim v_R$ and these waves are almost in phase and $\pi/2$ out of phase with v_C .

The same information can be represented on an

Argand diagram shown in figure 8.xx. As before the current is given by $i = I \cos \omega t$. The voltage across the



resistor $v_R = i \times R = IR \cos \omega t$ and we show the phasor representing v_R ($V_R = IR \angle 0$) lying along the real axis according to our convention. The voltage across the capacitor (v_{out}) lags the current by $\pi/2$ and is represented by the phasor $V_C = IX_C \sin \omega t$ which lies along the negative imaginary axis as shown. To find the phasor representing v_{in} we add the two phasors using the parallelogram rule as shown by the dot/dashed line. The amplitude of v_{in} is $(I^2/\omega^2 C^2 + I^2 R^2)^{1/2}$ while the phase difference between the input and output voltages is given by $\tan \phi = -\omega CR$. In phasor form this is $V_{in} = I (1/\omega^2 C^2 + R^2)^{1/2} \angle \arctan -\omega CR$, which leads

directly to (8.4)

8.5 The high pass filter

If we interchange the capacitor and resistor as shown in figure 8.5 it is easy to show that:

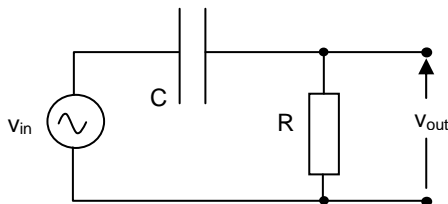


Figure 8.5

$$\frac{v_{out}}{v_{in}} = A = \frac{1}{1 + 1/j\omega CR} = \frac{1}{1 - j\omega_0/\omega} \quad (8.6)$$

Rationalising this complex expression gives

$$\left| \frac{v_{out}}{v_{in}} \right| = \frac{1}{(1 + \omega_0^2/\omega^2)^{1/2}} \quad (8.7)$$

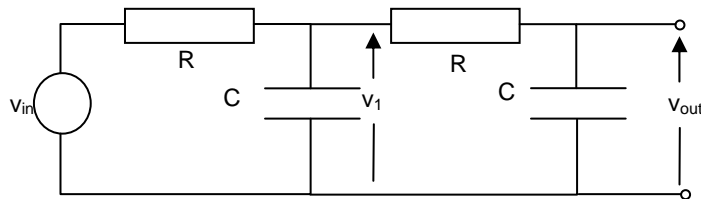
- At low frequencies where $\omega \ll \omega_0$ the gain $\sim \omega/\omega_0$, is small and increases with increasing ω
- At high frequencies the gain ~ 1 .

This behaviour is the opposite to that found for the low pass filter and not surprisingly this circuit is called a high pass filter. The phase shift is given by $\phi = \tan^{-1} 1/\omega CR$. At low frequency ($< \omega_0/10$) v_{out} leads v_{in} by $\pi/2$; at high frequencies ($> 10\omega_0$) v_{out} and v_{in} are in phase. At $\omega = \omega_0$ v_{out} leads v_{in} by $\pi/4$.

For a simple Java version of the low and high pass filters see:

https://www.st-andrews.ac.uk/~www_pa/Scots_Guide/experiment/lowpass/lpf.html

8.6 Cascaded low pass filters



If two low pass filters are cascaded the output voltage is not simply the product of two transfer functions. If you try to analyse such a circuit using phasors and an Argand diagram you will soon realise the shortcomings of that technique. This is

because the second filter acts as a load for the first. Although slightly more involved than anything presented until now it is possible to show that the gain is given by:

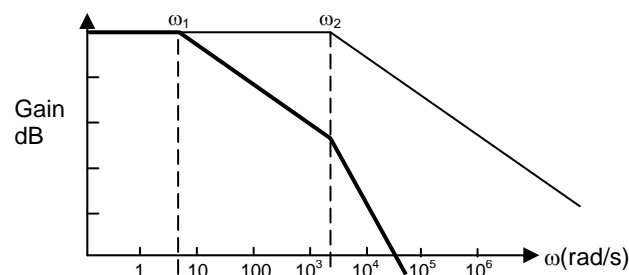
$$\frac{v_{out}}{v_{in}} = \frac{1}{(1 - \omega^2 C^2 R^2) + j3\omega CR}$$

When $\omega = 1/CR$, $\frac{v_{out}}{v_{in}} = \frac{1}{j3}$ and the amplitude of the gain is $1/3$ and the phase is $-\pi/2$

However, we shall later that a simple operational amplifier circuit called the unity gain buffer acts as an impedance matcher so that loading of the first filter by the second does not occur. For such an arrangement employing different values of R and C corresponding to different values of ω_0 ($= \omega_1$ and ω_2):

$$\frac{v_{out}}{v_{in}} = \frac{1}{(1 + j\omega/\omega_1)(1 + j\omega/\omega_2)}$$

where ω_1 and ω_2 are the corner frequencies of the two circuits. The figure shows the resulting Bode plot and also



indicates a useful property the log-log plots, namely that they can simply be added. Above ω_2 the roll-off is twice that of the single circuit i.e. -12 dB per octave.