$$\frac{\partial R}{\partial a_1}=0,\frac{\partial R}{\partial a_2}=0,\frac{\partial R}{\partial a_3}=0,\frac{\partial R}{\partial a_4}=0 \tag{4.44}$$

Equations (4.44) can be written as follows

$$\begin{aligned}
&2-\alpha_1-\alpha_2=0\\
&-\alpha_1 r_1-\alpha_2 r_2=0\\
&\frac{2}{3}-\alpha_1 r_1^{\,2}-\alpha_2 r_2^{\,2}=0\\
&-\alpha_1 r_1^{\,3}-\alpha_2 r_2^{\,3}=0
\end{aligned} \tag{4.45}$$

Solving the previous set of equations gives

$\alpha_1=1$, $\alpha_2=1$, $r_1=-1/\sqrt{3}=-0.5773502692$ and $r_2=1/\sqrt{3}=0.5773502692$.

The same procedure can be followed to calculate $\alpha_1,\alpha_2,.....,\alpha_n$ and $r_1,r_2,.....,r_n$ for any value of n. These values are calculated and shown in the table below for values of $n=1,2,3,4$.
For two-dimensional coordinates, numerical integrations are calculated using the following expression

$$\int_{-1}^{1}\int_{-1}^{1} f(r,s)drds=\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j f(r_i,s_j) \tag{4.46}$$

where the weight values and sampling points are shown in Table 2 and Figure 8 below.

Example 3

Calculate the following integration using the Gauss-Legendre numerical integration

$$I=\int_{-1}^{1}\int_{-1}^{1} r^2 e^s drds$$

From equation (4.46) and using the values in Table 2 and Figure 8 below

$$I=\int_{-1}^{1}\int_{-1}^{1} r^2 e^s drds=\sum_{i=1}^{2}\sum_{j=1}^{2}\alpha_i\alpha_j r_i^2 e^{s_j}$$

$$I = \alpha_1\alpha_1 r_1^2 e^{s_1} + \alpha_1\alpha_2 r_1^2 e^{s_2} + \alpha_2\alpha_1 r_2^2 e^{s_1} + \alpha_2\alpha_2 r_2^2 e^{s_2}$$

$$I = (-0.577350)^2 e^{(-0.577350)} + (-0.577350)^2 e^{(0.577350)} + (0.577350)^2 e^{(-0.577350)} + (0.577350)^2 e^{(0.577350)} = 1.562$$

The exact integration gives I=1.567.

Table2: Sampling points and weights in Gauss-Legendre numerical integration (interval -1 to +1). (Reference: Bathe, Finite Element Procedures, Prentice Hall)

n	r_i	α_i
1	0.0000000000	2.0000000000
2	$\pm$ 0.5773502692	1.0000000000
3	$\pm$ 0.7745966692	0.5555555556
	0.0000000000	0.8888888889
4	$\pm$ 0.8611363116	0.3478548451
	$\pm$ 0.3399810435	0.6521451548

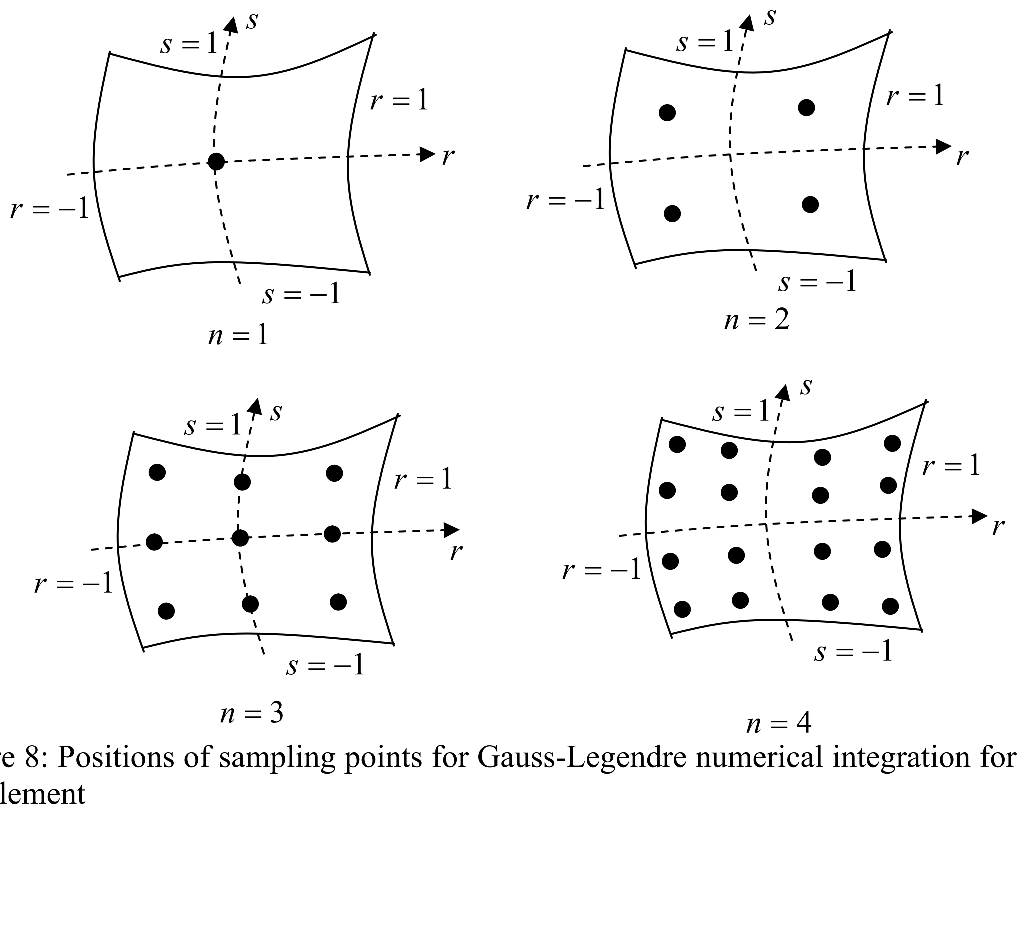

Figure 8: Positions of sampling points for Gauss-Legendre numerical integration for a 2D Element