

THE SPACETIME APPROACH TO QUANTUM MECHANICS*

James B. Hartle[†]

Department of Physics, University of California

Santa Barbara, CA 93106-9530 USA

Abstract

Feynman's sum-over-histories formulation of quantum mechanics is reviewed as an independent statement of quantum theory in spacetime form. It is different from the usual Schrödinger-Heisenberg formulation that utilizes states on spacelike surfaces because it assigns probabilities to different sets of alternatives. In a sum-over-histories formulation, alternatives at definite moments of time are more restricted than in usual quantum mechanics because they refer only to the coordinates in terms of which the histories are defined. However, in the context of the quantum mechanics of closed systems, sum-over-histories quantum mechanics can be generalized to deal with spacetime alternatives that are not "at definite moments of time". An example in field theory is the set of alternative ranges of values of a field averaged over a spacetime region. An example in particle mechanics is the set of the alternatives defined by whether a particle never crosses a fixed spacetime region or crosses it at least once. The general notion of a set of spacetime alternatives is a partition (coarse-graining) of the histories into an exhaustive set of exclusive classes. With this generalization the sum-over-histories formulation can be said to be in fully spacetime form with dynamics represented by path integrals over spacetime histories and alternatives defined as spacetime partitions of these histories. When restricted to alternatives at definite moments of times this generalization is equivalent to Schrödinger-Heisenberg quantum mechanics. However, the quantum mechanics of more general spacetime alternatives does not have an equivalent Schrödinger-Heisenberg formulation. We suggest that, in the quantum theory of gravity, the general notion of "observable" is supplied by diffeomorphism invariant partitions of spacetime metrics and matter field configurations. By generalizing the usual alternatives so as to put quantum theory in fully spacetime form we may be led to a covariant generalized quantum mechanics of spacetime free from the problem of time.

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[†]Electronic address: hartle@physics.ucsb.edu

I. INTRODUCTION

In 1948 Feynman [1], building on the work of Dirac [2], introduced his sum-over-histories formulation of quantum mechanics and with it the path integral that has proved a powerful tool in many branches of physics. It is possible to see the sum-over-histories as merely a technical tool — a useful device for computing certain amplitudes within the usual Schrödinger–Heisenberg formulation of quantum mechanics in terms of states on spacelike surfaces. That is not, however, how I think Feynman saw it. Rather, the sum-over-histories formulation of quantum mechanics can be regarded as an independent formulation of quantum theory. Its equivalence with the usual formulations is not automatic but rather a question whose answer may be different in different theories and for different physical systems. In this talk I shall review the current status of the sum-over-histories formulation as an independent statement of quantum theory. I shall argue that, viewed most fundamentally, it is *different* from the Schrödinger–Heisenberg formulations because the totality of alternatives to which it potentially assigns probabilities is different from that of Schrödinger–Heisenberg quantum mechanics. Sum-over-histories alternatives at definite moments of time are more restricted than in usual quantum mechanics because they refer only to the coordinates of the configuration space in which the paths are defined. By contrast, at a moment of time, Schrödinger–Heisenberg quantum mechanics has all the alternatives provided by transformation theory. However, even in non-relativistic quantum mechanics, the sum-over-histories formulation allows a generalization of the alternatives at definite moments of time to genuine spacetime alternatives that are not considered in usual Schrödinger–Heisenberg formulations. This generalization allows a more realistic description of everyday experiments. But, more importantly, it may be central for the construction of a quantum theory of gravity in which there is no well-defined notion of time. There, alternatives “at a moment of time” may be difficult to find, while spacetime alternatives may be the natural “observables” for which the theory makes predictions.

II. THE SPACETIME APPROACH TO QUANTUM MECHANICS

In the Schrödinger–Heisenberg quantum mechanics of particles or fields moving in a fixed background spacetime, the quantum dynamics of measured subsystems is formulated in terms of state vectors defined on spacelike surfaces that evolve unitarily in between measurements and are reduced at measurements. Unitary evolution is represented by

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi\rangle \quad (2.1)$$

where H is the Hamiltonian and $|\psi\rangle$ is the state at $t = 0$. From (2.1) we could calculate the transition amplitude between coordinates q' at time t' to coordinates q'' at time t'' , that is an equivalent summary of unitary evolution, *viz.*

$$\langle q''t''|q't'\rangle = \langle q''|e^{-iH(t''-t')/\hbar}|q'\rangle. \quad (2.2)$$

(Coordinate indices, which may refer to either particles or fields, are often omitted to keep the notation compact). In a sum-over-histories formulation of quantum mechanics such amplitudes, are specified *directly* as path integrals

$$\langle q''t''|q't'\rangle = \int_{[q',q'']} \delta q \exp(iS[q(\tau)]/\hbar). \quad (2.3)$$

Here, $S[q(\tau)]$ is the action functional corresponding to the Hamiltonian H and the sum is over paths $q(t)$ that start at q' at time t' , end at q'' at time t'' , and are single-valued functions of time. In an abbreviated notation, we may write

$$\int \delta q' \exp(iS[q(\tau)]/\hbar)|\psi\rangle \quad (2.4)$$

for the state vector $|\psi(t)\rangle$ that is evolved by the propagator (2.3). More explicitly, (2.4) stands for the state vector $|\psi(t)\rangle$ whose representative wave function $\psi(q, t) = \langle q|\psi(t)\rangle$ is

$$\psi(q, t) = \int dq' \left(\int_{[q',q]} \delta q \exp(iS[q(\tau)]/\hbar) \right) \psi(q', t'). \quad (2.5)$$

In this way, quantum dynamics corresponding to unitary evolution is cast into manifestly spacetime form involving spacetime histories directly. This is an important advantage in dealing with spacetime symmetries such as Lorentz invariance.

Unitary evolution, however, is not the only law by which the state vector evolves in quantum mechanics. In the usual discussion, at an ideal measurement that disturbs the

measured system as little as possible, the state vector is instantaneously “reduced” according to the “second law of evolution”

$$|\psi(t)\rangle \longrightarrow \frac{P_\alpha |\psi(t)\rangle}{\|P_\alpha |\psi(t)\rangle\|} \quad (2.6)$$

where P_α is the projection operator on the subspace corresponding to the outcome of the measurement and $\|\cdot\|$ denotes the norm of a vector in Hilbert space. This “second law of evolution” may not be needed to calculate the transition probabilities in scattering experiments but it is essential for calculating the probabilities of the *sequences* of observations that define the histories of everyday life such as the orbit of the earth around the sun. It is every bit as essential for the prediction of realistic probabilities as is unitary evolution.

Using the two laws of evolution, the joint probability for a sequence of measured outcomes $\alpha_1, \dots, \alpha_n$ at times t_1, \dots, t_n may be calculated. It finds its most compact expression in the Heisenberg picture:

$$\left\| P_{\alpha_n}^n(t_n) P_{\alpha_{n-1}}^{n-1}(t_{n-1}) \cdots P_{\alpha_1}^1(t_1) |\psi\rangle \right\|^2. \quad (2.7)$$

Here, $\{P_{\alpha_k}^k(t_k)\}$ is an exhaustive set of orthogonal Heisenberg picture projections representing the alternatives α_k in the set k at time t_k . For example, the set of alternatives might be an exhaustive set of alternative regions $\Delta_{\alpha_k}^k$ of the coordinates q at time t_k .

The second law of evolution can be expressed simply in sum-over-histories form if attention is restricted to alternatives defined by sequences of configuration space regions $\{\Delta_{\alpha_1}^1\}, \{\Delta_{\alpha_2}^2\}, \dots$ at times t_1, \dots, t_n [4, 5]. The joint probability that the system passes through the particular sequence of regions $\alpha = (\alpha_1, \dots, \alpha_n)$ is

$$\left\| \int_{c_\alpha} \delta q \exp(iS[q(\tau)]/\hbar) |\psi\rangle \right\|^2 \quad (2.8)$$

where the path integral is over the class of paths c_α that pass through the region $\Delta_{\alpha_1}^1$ at time t_1 , $\Delta_{\alpha_2}^2$ at time t_2 , etc. Eq. (2.8) is a unified expression for the two laws of evolution in quantum mechanics.

Eq. (2.8) is equivalent to (2.7) for alternatives defined by exhaustive sets of exclusive regions of configuration space at definite moments of time. In this case the Schrödinger–Heisenberg formulation of quantum mechanics and the sum-over-histories formulation coincide. But they are not *fully* equivalent because the effect of projections onto ranges of

momentum, for example, cannot be represented as restrictions on a configuration space path integral as in (2.8). Probabilities for momentum measurements *can* be predicted using path integrals, but only approximately, by modeling a time of flight determination of velocity in configuration space terms [21]. The sum-over-histories formulation of quantum mechanics therefore deals directly and exactly with a more restricted class of alternatives at sequences of moments of time than is available from the transformation theory of the Schrödinger–Heisenberg formulation.

III. SPACETIME ALTERNATIVES

As we have presented it so far, the sum-over-histories formulation of quantum mechanics is not in fully spacetime form. The use of path integrals over spacetime histories has put the dynamics corresponding to unitary evolution into spacetime form. But the alternatives to which the theory assigns probabilities are not general spacetime alternatives. They have been restricted to sequences of alternative regions of configuration space at definite moments of time. More general spacetime alternatives are easy to imagine. We mention two: Consider the quantum mechanics of a particle in which the histories are paths in spacetime. Fix a spacetime region R with extent both in space and time (Figure 1). A given particle path may never cross R or, alternatively, it may cross R sometime, perhaps more than once. These are an exhaustive set of spacetime alternatives for the particle that are not “at a moment of time”. A second example is provided by field averages over spacetime regions with extent both in space and time such as were considered by Bohr and Rosenfeld [6] in their discussion of the measurability of the electromagnetic field. An exhaustive set of ranges of such average values of a field is an example of a set of spacetime alternatives in field theory. Such spacetime alternatives are not directly assigned probabilities in Schrödinger–Heisenberg quantum mechanics because they are not “at a moment in time”.

In his 1948 paper, Feynmann discussed alternatives defined with respect to spacetime regions such as we described above. In particular, he offered a sum-over-histories definition of the probability that “if an ideal measurement is performed to determine whether a particle has a path lying in a region of spacetime ... the result will be affirmative”. However, that discussion, as well as more recent ones [4, 7–9, 29, 38] were incomplete because they did

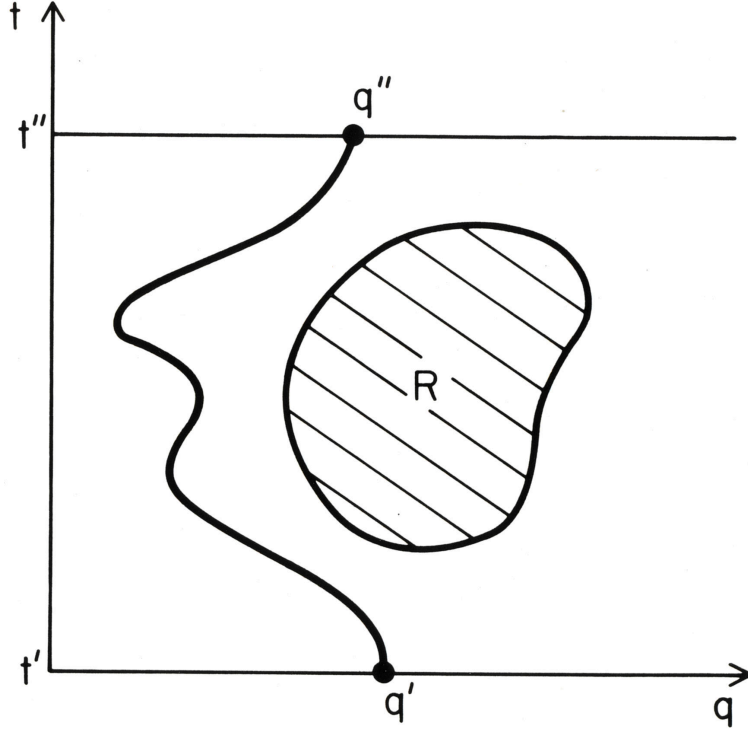


FIG. 1: Spacetime Alternatives defined by a spacetime region. The figure shows a spacetime region R with extent in both space and time. The paths of a non-relativistic particle between q' at time t' and q'' at time t'' may be divided into two classes: First, the class of paths that never cross the region R (illustrated). Second, the class of paths that cross R sometime. These two classes constitute an exhaustive set of alternatives for the particle that are not “at a moment of time”.

not specify clearly what such an ideal measurement consisted of or what was to replace the “second law of evolution”, (2.6), following its completion. “I have not been able to find a precise definition” Feynman said [1]. Only recently has it become clear how to generalize sum-over-histories quantum mechanics to predict probabilities for such spacetime alternatives within the quantum mechanics of closed systems in which the notion of “measurement” does not play a fundamental role [3, 10–12]. I shall discuss this generalization, but first we must review a bit of the quantum mechanics of closed systems.

IV. THE QUANTUM MECHANICS OF CLOSED SYSTEMS

The most general objective of quantum theory is to predict the probabilities of individual histories in an exhaustive set of alternative, coarse-grained histories of a closed system. A characteristic feature of a quantum-mechanical theory is that not every set of histories that may be described can be assigned probabilities because of quantum-mechanical interference between the individual histories in the set. Nowhere is this more clearly illustrated than in the two-slit experiment (Figure 2). In the usual discussion, if we have not measured which slit the electron went through on its way to being detected at the screen, then we are not permitted to assign probabilities to the alternative histories in which it passed through the upper or lower slit. It would be inconsistent to do so since the correct probability sum rules would not be satisfied. Because of interference, the probability to arrive at a point y on the screen is not the sum of the probabilities to arrive at y going through the upper and the lower slit:

$$|\psi_L(y) + \psi_U(y)|^2 \neq |\psi_L(y)|^2 + |\psi_U(y)|^2. \quad (4.1)$$

In quantum mechanics a rule is needed to determine which sets of histories can be assigned probabilities and then what those probabilities are.

In the “Copenhagen” quantum mechanics of measured subsystems probabilities can be assigned to alternative histories that have been *measured*. In the two slit experiment, for example, if we *have* measured which slit the electron went through, then interference is destroyed, the sum rule obeyed and we *can* consistently assign probabilities to the alternative histories in which the electron passed through the upper or lower slit.

In the quantum mechanics of closed systems, containing both observer and observed, measuring apparatus and measured subsystem, the above rule is but a special case of a more general one of much wider applicability [11, 13–15]. Probabilities can be assigned to the individual members of a set of alternative coarse-grained histories of a closed system when there is negligible quantum-mechanical interference between these histories as a consequence of the system’s initial condition and dynamics. Such sets of histories are said to *decohere*.

To describe more precisely what is meant by decoherence, consider for simplicity, a pure initial state $|\psi\rangle$ and a set of histories defined by sets of alternatives at definite moments of

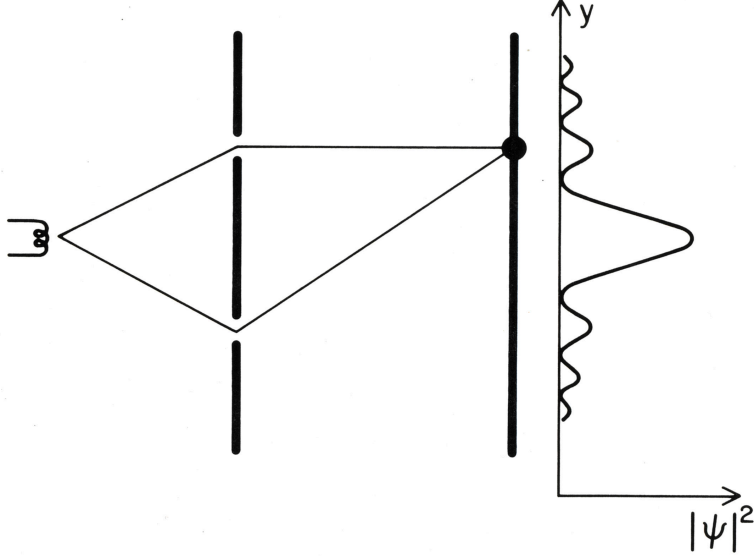


FIG. 2: The two-slit experiment. An electron gun at left emits an electron traveling towards a screen with two slits, its progress in space recapitulating its evolution in time. When precise detections are made of an ensemble of such electrons at the screen it is not possible, because of interference, to assign a probability to the alternatives of whether an individual electron went through the upper slit or the lower slit. However, if the electron interacts with apparatus that measures which slit it passed through, then these alternatives decohere and probabilities can be assigned.

time t_1, \dots, t_n . The alternatives at the moment of time t_k are described by an exhaustive set of exclusive Schrödinger picture projection operators $\{P_{\alpha_k}^k\}$ satisfying

$$\sum_{\alpha_k} P_{\alpha_k}^k = 1, \quad P_{\alpha_k}^k P_{\beta_k}^k = \delta_{\alpha_k \beta_k} P_{\alpha_k}^k. \quad (4.2)$$

In this notation k denotes the set of alternatives at time t_k (*e.g.* a set of position ranges or a set of momentum ranges, etc.) and α_k the particular alternative. The alternatives are *fine-grained* if the P 's project onto the one-dimensional subspaces defined by a complete set of states and otherwise are *coarse-grained*.

The sequences of alternatives at definite moments of time $0 < t_1 < t_2 < \dots < t_n < T$ define a set of coarse-grained alternative histories on a time interval $[0, T]$. The individual histories correspond to particular sequences $\alpha = (\alpha_1, \dots, \alpha_n)$ which are represented by the

corresponding chains of projection operators interrupted by unitary evolution

$$C_\alpha = e^{-iH(T-t_n)/\hbar} P_{\alpha_n}^n e^{-iH(t_n-t_{n-1})/\hbar} P_{\alpha_{n-1}}^{n-1} \dots e^{-iH(t_2-t_1)/\hbar} P_{\alpha_1}^1 e^{-iHt_1/\hbar}. \quad (4.3)$$

These may be written somewhat more compactly using Heisenberg picture projections as

$$C_\alpha = e^{-iHT/\hbar} P_{\alpha_n}^n(t_n) \dots P_{\alpha_1}^1(t_1), \quad (4.4)$$

where

$$P_{\alpha_k}^k(t) = e^{iHt/\hbar} P_{\alpha_k}^k e^{-iHt/\hbar}. \quad (4.5)$$

Since $\sum_\alpha C_\alpha = e^{iHT/\hbar}$ as a consequence of (4.2) and (4.3), the evolution of the initial state $|\psi\rangle$ may be resolved into branches, $|\psi_\alpha\rangle$, corresponding to the individual histories

$$e^{-iHT/\hbar} |\psi\rangle = \sum_\alpha C_\alpha |\psi\rangle \equiv \sum_\alpha |\psi_\alpha\rangle. \quad (4.6)$$

A set of histories decoheres¹ when the individual branches are essentially orthogonal

$$\langle \psi_{\alpha'} | \psi_\alpha \rangle \approx 0, \quad \alpha' \neq \alpha. \quad (4.7)$$

The probabilities of the individual histories in such a decoherent set are the square of the norms of the corresponding branches

$$p(\alpha) = \|\psi_\alpha\|^2. \quad (4.8)$$

Eq. (4.8) is a consistent assignment of probabilities to a decoherent set of histories because decoherence implies that the probability sum rules are satisfied in their most general form. To give a simple example, consider a set of histories defined by alternatives at just two moments of time t_1 and t_2 and the probability sum rule

$$\sum_{\alpha_1} p(\alpha_2, \alpha_1) = p(\alpha_2). \quad (4.9)$$

¹ The term “decoherence” is used in several different ways in the literature. We have followed our earlier work [15] in using the term to refer to a property of a set of coarse-grained *histories* of a closed system namely the absence of interference between individual histories in the set at a level to ensure the consistency of the probability sum rules. There are several different measures of decoherence [30]. The condition (4.7) is called the “medium decoherence condition”. In this simplified presentation “decoherence” therefore means the medium decoherence of histories. Similar conditions are called “consistency conditions” [13, 14] or “no-interference conditions” [11]. The term decoherence has also been used to refer to the approach to diagonality of a reduced density matrix in a particular basis. The decoherence of density matrices is not the same as the decoherence of histories in general but the two ideas can be related in special models.

For the left hand side of (4.9) we may write

$$\begin{aligned}
\sum_{\alpha_1} p(\alpha_2, \alpha_1) &= \sum_{\alpha_1} \langle \psi | P_{\alpha_1}^1(t_1) P_{\alpha_2}^2(t_2) \cdot P_{\alpha_2}^2(t_2) P_{\alpha_1}^1(t_1) | \psi \rangle \\
&= \sum_{\alpha'_1 \alpha_1} \langle \psi | P_{\alpha'_1}^1(t_1) P_{\alpha_2}^2(t_2) \cdot P_{\alpha_2}^2(t_2) P_{\alpha_1}^1(t_1) | \psi \rangle \\
&= \langle \psi | P_{\alpha_2}^2(t_2) \cdot P_{\alpha_2}^2(t_2) | \psi \rangle = p(\alpha_2) .
\end{aligned} \tag{4.10}$$

The first and last equality are the definition (4.8). The second equality is true because of the decoherence condition (4.7), and the third because of (4.2). Thus, decoherence implies the probability sum rules needed for a consistent assignment of probabilities.

Measured alternatives decohere but an alternative need not be a participant in a measurement situation in order to decohere. In cosmology, quantum theory predicts the probabilities of alternative sizes of density fluctuations one minute after the big bang, in a universe in which these alternatives decohere, whether or not anything like a measurement was carried out on them and certainly whether or not there was an observer around to do it. Decoherence is thus a more precise, more general, and more observer-independent notion than measurement and replaces it in the quantum mechanics of closed systems as the criterion determining which sets of histories can be assigned probabilities.

V. A GENERALIZED SUM-OVER-HISTORIES QUANTUM MECHANICS FOR SPACETIME ALTERNATIVES

Building on the sum-over-histories ideas, the quantum mechanics of alternatives at definite moments of time that was described in the previous section may be generalized to deal with the spacetime alternatives described in Section III. The result is a quantum framework for prediction with dynamics and alternatives fully in spacetime form. We shall describe this generalization for the case of a configuration space spanned by coordinates, q^i , and assume a pure initial state. There is nothing essential about these restrictions. The generalization to an initial density matrix requires only a modest expansion of the formalism and the coordinates q^i could be the values of fields $\phi(\vec{x})$ at each point in space. The important assumption is that there is a fixed background spacetime that supplies a well defined notion of time. We follow the discussion in [3].

The most refined possible description of a closed system are its fine-grained histories.

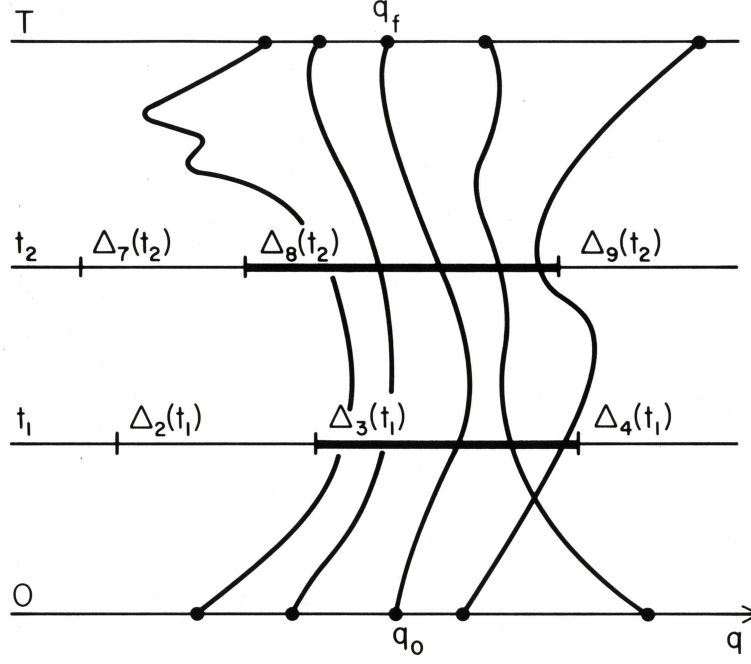


FIG. 3: Coarse graining by regions of configuration space at successive moments of time. The figure shows a spacetime that is a product of a one-dimensional configuration space (q) and the time interval $[0, T]$. At times t_1 and t_2 the configuration space is divided into exhaustive sets of non-overlapping intervals: $\{\Delta_{\alpha_1}(t_1)\}$ at time t_1 , and $\{\Delta_{\alpha_2}(t_2)\}$ at time t_2 (written in the text as $\{\Delta_{\alpha_1}^1\}$, $\{\Delta_{\alpha_2}^2\}$, etc). Some of these intervals are illustrated. The fine-grained histories are the paths which pass between $t = 0$ and $t = T$. Because the paths are assumed to be single-valued in time, the set of fine-grained histories may be partitioned into two classes according to which intervals they pass through at times t_1 and t_2 . The figure illustrates a few representative paths in the class which pass through region $\Delta_3(t_1)$ at time t_1 and region $\Delta_8(t_2)$ at time t_2 .

These are the paths $q^i(t)$ that are single-valued functions of the time. Partitions of these fine-grained histories into exhaustive sets of exclusive classes c_α yield sets of coarse-grained histories. An individual coarse-grained history c_α is thus a class of fine-grained histories. Exhaustive sets of alternative coarse-grained histories $\{c_\alpha\}$ are the general notion of “observable” for which quantum theory predicts probabilities when the set of coarse-grained histories is decoherent.

Sequences of alternative ranges of coordinates $\{\Delta_{\alpha_1}^1\}, \{\Delta_{\alpha_2}^2\}, \dots, \{\Delta_{\alpha_n}^n\}$ at, say, times t_1, \dots, t_n define one kind of partition of the paths $q^i(t)$. An individual coarse-grained history

corresponding to the sequence of alternatives $\alpha = (\alpha_1, \dots, \alpha_n)$ is the class of paths c_α that thread the region $\Delta_{\alpha_1}^1$ at time t_1 , $\Delta_{\alpha_2}^2$ at time t_2 , etc. (Figure 3) These correspond to sequences of the familiar “observables” at definite moments of time. However, much more general partitions are possible. For example, following the discussion of Section III, paths may be partitioned into two classes according to their behavior with respect to a spacetime region R . One class, c_0 , consists of all paths that never intersect R and the other, c_1 , consists of all paths that intersect R at least once. These classes are exclusive and together they are exhaustive. They constitute a coarse-grained set of histories.

The most general notion of coarse-grained histories is a partition by values of functionals of histories [4, 37]. Consider, for example, just a single functional $F[q(\tau)]$ and a set of ranges $\{\Delta_\alpha\}$ of the real line. The class c_α consists of all paths $q^i(t)$ such that $F[q(\tau)]$ lies in the range Δ_α . We could partition the paths, for example, by the value of some particular coordinate, q^k , averaged over a time interval, that is, by the functional

$$F[q(\tau)] = \frac{1}{T} \int_0^T dt q^k(t). \quad (5.1)$$

In this way we could deal with the average values of fields over spacetime regions whose importance was stressed by Bohr and Rosenfeld [6]. In a field theory with a spinor field $\psi(x)$ we could partition the field histories by the values of currents, *e.g.* $\psi^\dagger(\vec{x}, t)\psi(\vec{x}, t)$. In this way the theory can incorporate observables associated with spin. In the theory of a non-relativistic particle we could partition the paths by the value of the position *difference* between two times separated by a time interval T . In the limit that T becomes large but still short compared to dynamical time scales of any interaction these partitions define momentum alternatives determined by the time of flight [21]. In this way probabilities for momenta can be predicted by the theory.

Sums over the fine-grained histories contained in a coarse-grained history define the branch of the initial state corresponding to that history. To make this explicit, consider histories on the time interval $[0, T]$ and suppose that the fine-grained histories on this interval are partitioned into an exhaustive set of exclusive classes c_α . We define branches of the initial state and class operators C_α corresponding to each class by

$$|\psi_\alpha\rangle \equiv C_\alpha|\psi\rangle = \int_{c_\alpha} \delta q \exp(iS[q(\tau)]/\hbar) |\psi\rangle \quad (5.2a)$$

where the integral is over all paths in the class c_α . We are using the same abbreviated notation as in (2.4). More explicitly, (5.2a) stands for

$$\psi_\alpha(q, T) = \langle q | C_\alpha | \psi \rangle = \int dq' \int_{[q' c_\alpha q]} \delta q \exp(iS[q(\tau)]/\hbar) \psi(q', 0). \quad (5.2b)$$

Evidently, since the sum over all paths just gives unitary evolution as in (2.4), we have

$$\sum_\alpha C_\alpha = e^{-iHT/\hbar}. \quad (5.3)$$

Decoherence and probabilities are defined as before. The set of coarse-grained histories $\{c_\alpha\}$ decoheres if

$$\langle \psi_{\alpha'} | \psi_\alpha \rangle \approx 0 \quad \alpha' \neq \alpha \quad (5.4)$$

and the probability of an individual history c_α in a decoherent coarse-grained set is

$$p(c_\alpha) = \|\psi_\alpha\|^2. \quad (5.5)$$

The important point about this construction is that the probability sum rules, which are the obstacle to consistently assigning probabilities in quantum mechanics, are satisfied for a decoherent set of coarse-grained histories as a consequence of (5.4). To see this consider a partition of the $\{c_\alpha\}$ into an exhaustive set of exclusive classes yielding a coarser-grained partition of the fine-grained histories $\{\bar{c}_\beta\}$. The most general form of the probability sum rules is

$$p(\bar{c}_\beta) = \sum_{\alpha \in \beta} p(c_\alpha) \quad (5.6)$$

where the sum is over α such that c_α included in \bar{c}_β . This is easily seen to be satisfied as a consequence of (5.2) and (5.4). From the linearity of (5.2) that reflects the principle of superposition it follows that

$$\bar{C}_\beta = \sum_{\alpha \in \beta} C_\alpha. \quad (5.7)$$

But then a repetition of the argument that led to (4.9) shows the validity of the more general sum rule (5.6). In this way sum-over-histories ideas can be used to formulate a generalized quantum mechanics of closed systems that is fully in spacetime form with dynamics summarized by path integrals over alternatives that are general partitions of spacetime histories.

VI. COMPARISON WITH SCHRÖDINGER-HEISENBERG QUANTUM MECHANICS

We mentioned that, for alternatives at moments of time, sum-over-histories quantum mechanics deals directly only with configuration space alternatives in contrast to Schrödinger-Heisenberg quantum mechanics which utilizes all the possibilities of transformation theory. The sum-over-histories formulation is thus more restricted in its alternatives at moments of time. Configuration space variables have a preferred place in the formalism.

However, in the preceeding section we have shown how sum-over-histories ideas can be used to formulate a quantum mechanics that deals with spacetime alternatives that are much more general than those “at moments of time” with which the Schrödinger-Heisenberg formulation is concerned. This generalized spacetime quantum mechanics cannot be reformulated in terms of the two laws of evolution of the Schrödinger-Heisenberg formulation. If it could be so formulated, the class operators C_α defined by (5.2) for each history in a spacetime coarse graining could be represented as a chain of projections of the form (4.4) or perhaps a continuous product of such projections one for each time. Then one could describe the evolution completely in terms of unitary evolution interrupted (perhaps continuously in time) by reductions of the state vectors.

However, the class operators of a spacetime coarse graining cannot, in general, be represented as products of projections, even continuous ones. Consider, by way of example, the coarse graining of the paths of a particle by their behavior with respect to a spacetime region R that was discussed earlier. The class operator C_0 for the class of paths that never cross R *can* be represented as a continuous chain of projections on the region of q outside of R at each time. However, the class operator C_1 for the class of paths that crosses R sometime cannot be so represented because at each time the particle could be either inside R or outside it. The class operator C_1 can be represented formally as a *sum* of continuous chain of projections

$$C_\alpha = \sum_{\alpha(t) \in c_\alpha} \prod_t P_{\alpha(t)}^{k(t)}(t). \quad (6.1)$$

However, a quantum mechanics of alternatives represented by class operators that are *sums* of chains of projections already constitutes a generalization of familiar quantum mechanics [10, 32] although a very natural one.

The probabilities of spacetime coarse-grained histories thus cannot be expressed in terms of a unitarily evolving state vector that is “reduced” at various moments of time. The reason, it should be stressed, does not lie in the use of path integrals versus operators. Indeed, as (5.2) shows there is a correspondence between path integrals and operators and operator techniques provide the most convenient way of defining the path integrals on the right hand side of (5.2) [3]. Rather, it is the *spacetime* nature of the alternatives that does not allow meaningful notion of state of the system at a moment of time.

This generalized sum-over-histories quantum mechanics is thus not equivalent to the Schrödinger-Heisenberg formulation because the two formulations deal with different alternatives. Sums of continuous products like (6.1) could be taken as the starting point of a yet more generalized quantum mechanics that would contain all the alternatives of both. Such a generalization presents interesting mathematical problems. In the meantime, to the extent our experience can be expressed in terms of spacetime alternatives, nothing seems to be lost by a restriction to these and much would seem to be gained from a more realistic description of alternatives are extended over time. In the next section I shall argue that the spacetime approach to quantum mechanics has definite advantages in the quantum theory of spacetime where there is no well defined notion of “at a moment of time”.

VII. A GENERALIZED QUANTUM MECHANICS OF SPACETIME

The nature of the “observables” to which a quantum theory of spacetime geometry assigns probabilities has always been something of a puzzle in quantum gravity. We cannot straightforwardly and covariantly define alternatives at a moment of time because there is no fixed notion of time. In a theory where the geometry of spacetime fluctuates quantum-mechanically, there is no fixed interval between spacetime points and not even a fixed notion of whether that interval is timelike or spacelike. Put differently, there is no covariant choice of geometrical variable to play the role of “ t ” in the predictive framework summarized by (4.4), (4.7), and (4.8). In essence, there is a conflict between the diffeomorphism invariance of spacetime theories of gravity and the requirements of Schrödinger-Heisenberg quantum mechanics. This is called the “problem of time” in quantum gravity. For recent critical surveys of various approaches to its resolution see [17–19, 36].

One thread of thought is that alternatives in quantum gravity should be represented by operators that commute with the constraints implied by diffeomorphism invariance. However, time reparametrizations are among the diffeomorphisms and for such theories the constraints generate the dynamics. Restricting the observables to operators that commute with the constraints corresponds classically to observables that are constants of the motion. This is a very restricted class of observables!²

The spacetime approach to quantum mechanics provides a different resolution to the problem of time. In a quantum theory of spacetime, the fine-grained histories are the possible four-dimensional metrics and matter field configurations on a fixed manifold M (in the simplest case). Quantum gravitational dynamics can be expressed in spacetime form through sums of $\exp(iS[g, \phi]/\hbar)$ over metrics and matter fields on M , where S is the action for spacetime and matter. The use of sums-over-histories to formulate a covariant quantum gravitational dynamics has been extensively investigated.³ However, the spacetime approach to quantum mechanics can also be used to specify the alternatives to which a quantum theory of spacetime assigns probabilities in a covariant way.

A generalized sum-over-histories quantum mechanics for spacetime can be constructed in which the fine-grained histories are the four-dimensional metrics and matter field configurations as discussed above [10, 37]. Allowed coarse grainings are partitions of these fine-grained histories into exhaustive sets of *diffeomorphism invariant* classes $\{c_\alpha\}$. These diffeomorphism invariant classes are the analogs of the spacetime coarse grainings we have discussed for quantum mechanics in fixed background geometries. They supply a very broad class of generally covariant alternatives to which, when decoherent, a quantum theory of spacetime will assign probabilities.

Many examples of diffeomorphism invariant coarse grainings could be given but in the limited space available I will confine myself to just two. First, we consider the probability that a closed cosmology reaches a maximum spatial volume greater than, say, V_0 . This question can be given a precise meaning by partitioning the class of all cosmological four-

² However, argued to be sufficient by some, Rovelli [33].

³ It has been investigated in a literature far too large to be cited here. However, some representative and important early papers are those of Misner [22], Leutwyler [34], DeWitt [23], Fradkin and Vilkovisky [35], Faddeev and Popov [24], Hawking [25], Teitelboim [26], and Polyakov [27]. There are many other important ones.

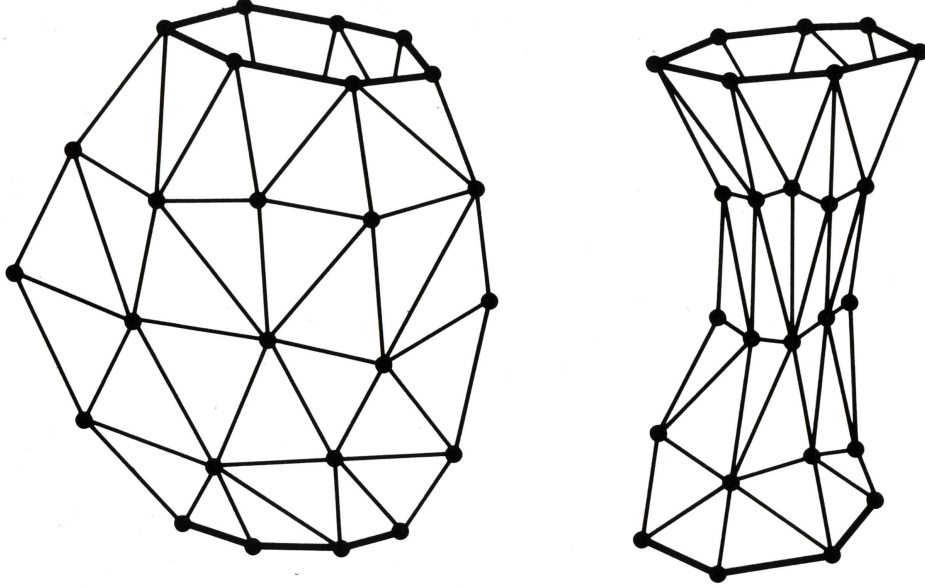


FIG. 4: Simplicial Geometries. Two-dimensional surfaces can be made up by joining together flat triangles to form a simplicial manifold. A geometry of the surface is specified by an assignment of squared edge-lengths to the triangles. The figure shows two different geometries obtained by a different assignment of squared edge-lengths to the same simplicial manifold. The generalization of these ideas to four dimensions and Lorentz signature gives the natural lattice version of general relativity — the Regge calculus. In a sum-over-histories quantum theory of simplicial spacetimes, sums over geometries are represented by integrals over the squared edge-lengths. Diffeomorphism invariant alternatives can be defined by partitioning the space of allowed squared edge-lengths into exhaustive sets of exclusive regions. For example, one could partition closed cosmological geometries into the class that has no simplicial spacelike three surface greater than a certain volume and the class that has at least one such surface. In a given simplicial manifold it is possible to enumerate all three surfaces and identify the regions in the space of squared edge-lengths to which each class corresponds.

metrics into the class that have at least one spacelike three-surface with a total volume greater than V_0 , and the class of three metrics that have no such three-surface. This is clearly an exhaustive partition into two exclusive classes that are each diffeomorphism invariant. Equally clearly the specification of these alternatives involves no preferred notion time.

As a second example, we consider the natural lattice formulation of general relativity —

the Regge calculus [28, 31]. A two-dimensional surface can be constructed from triangles (Figure 4). The topology of the surface is specified by how the triangles are joined together. The geometry of the surface is specified by an assignment of the squared edge-lengths of the triangles and a flat geometry to their interior. Similarly, four-dimensional Lorentzian geometries can be constructed out of four-simplices with squared edge-lengths that may be positive or negative. The space of geometries is parametrized by the n_1 squared edge-lengths s^1, \dots, s^{n_1} consistent with the analogs of triangle inequalities. A point in this space of squared edge-lengths is a fine-grained history. A partition of this space into regions that are invariant under the symmetries of the simplicial net provides a very general class of coarse grainings for these lattice geometries that does not require a preferred notion of time for its specification.

Beyond these examples, however, diffeomorphism invariant partitions of metrics and field configurations supply the *most general* notion of alternative in quantum theory that is describable in spacetime terms. Every property of the universe whose probability we may seek to calculate corresponds to some such coarse-graining, namely the partition of all four-metrics and matter field configurations into the class in which the property is true and the class in which it is false. If we cannot distinguish which fine-grained histories have the property and which do not then the property is not well defined.

To complete the construction of a generalized quantum mechanics a notion of decoherence must be specified for these coarse grained histories. I shall only indicate how to do this schematically. More details can be found in [37]. Branches, corresponding to individual coarse-grained histories in a set $\{c_\alpha\}$, can be represented as wave functions $\Psi_\alpha[h, \chi]$ on the superspace of spatial three metrics, $h_{ij}(\mathbf{x})$, and spatial matter field configurations, $\chi(\mathbf{x})$. These are defined in analogy to (5.2) or, more explicitly, (2.5)

$$\Psi_\alpha[h, \chi] = \int \delta h' \delta \chi' \int_{[(h', \chi'), c_\alpha, (h, \chi)]} \delta g \delta \phi \exp\left(iS[g, \phi]/\hbar\right) \circ \Psi[h', \chi'] . \quad (7.1)$$

In this expression $\Psi[h, \chi]$ is the wave function representing the initial condition of the universe, say, the “no-boundary” wave function [16]. The integral is over metrics and fields in the class c_α on a manifold M with two boundaries. On one boundary metrics and fields match the arguments (h, χ) of the branch wave function. On the other boundary they match the arguments (h', χ') of the initial condition. Much remains to be spelled out to make such

a construction concrete, not least the details of the measure and the product \circ with which the initial condition is attached to the functional integral and more details are in [37]. The important point is that with these branches one can define a decoherence condition for coarse-grained histories analogous to (5.4) and probabilities for the individual histories in decoherent sets by expressions analogous to (5.5). The quantum mechanics of spacetime is thus cast into a generally covariant fully spacetime form free from the problem of time.

How is the usual formulation of quantum mechanics with its preferred time variables connected with this generalization that requires no such preferred time? The answer is to be found by examining the origin of the classical spacetime of everyday experience. Classical spacetime is not a general feature of every state in a quantum theory of gravity. We expect classically behaving spacetime only for particular states and then only for coarse-grainings that define geometry well above the Planck scale. For such states and coarse-grainings the integral over metrics in (7.1) may be carried out by steepest descents. Suppose, in the simplest case, that only a single classical geometry \hat{g} dominates the sum. The remaining integrals over matter fields are equivalent to those of a field theory in the fixed background geometry of \hat{g} . The quantum mechanics of matter fields thus inherits its notions of time from the timelike directions of the classical background \hat{g} . It could be in this way that the familiar Hamiltonian formulation of quantum mechanics, with its preferred time(s), emerges as an approximation appropriate to the existence of an approximately quasiclassical spacetime in a more general, covariant, spacetime, formulation of quantum theory that is free from the problem of time.

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