Eigenvalues and Eigenvectors of the Matrix

Given matrix:

$$A = \begin{pmatrix} 2 & i \\ -i & 3 \end{pmatrix}$$

Eigenvalues

1. **Finding the characteristic polynomial:**

$$\det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & i \\ -i & 3-\lambda \end{pmatrix}$$

$$= (2-\lambda)(3-\lambda) - (-i \cdot i)$$

Since $i \cdot (-i) = 1$:

$$= (2-\lambda)(3-\lambda) - 1$$

Expanding and simplifying:

$$(2-\lambda)(3-\lambda) - 1$$

$$= 6 - 5\lambda + \lambda^2 - 1$$

$$= \lambda^2 - 5\lambda + 5$$

2. **Solving the quadratic equation:**

$$\lambda^2 - 5\lambda + 5 = 0$$

Using the quadratic formula:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $a = 1$, $b = -5$, and $c = 5$:

$$\lambda = \frac{5 \pm \sqrt{25 - 20}}{2}$$

$$= \frac{5 \pm \sqrt{5}}{2}$$

Thus, the eigenvalues are:

$$\lambda_1 = \frac{5 + \sqrt{5}}{2}$$

$$\lambda_2 = \frac{5 - \sqrt{5}}{2}$$

Eigenvectors

For $\lambda_1 = \frac{5+\sqrt{5}}{2}$

1. **Compute $A - \lambda_1 I$:**

$$A - \lambda_1 I = \begin{pmatrix} 2 - \frac{5+\sqrt{5}}{2} & i \\ -i & 3 - \frac{5+\sqrt{5}}{2} \end{pmatrix}$$

Simplifying:

$$2 - \frac{5+\sqrt{5}}{2} = \frac{4-(5+\sqrt{5})}{2} = \frac{-1-\sqrt{5}}{2}$$

$$3 - \frac{5+\sqrt{5}}{2} = \frac{6-(5+\sqrt{5})}{2} = \frac{1-\sqrt{5}}{2}$$

So:

$$A - \lambda_1 I = \begin{pmatrix} \frac{-1-\sqrt{5}}{2} & i \\ -i & \frac{1-\sqrt{5}}{2} \end{pmatrix}$$

2. **Solve $(A - \lambda_1 I)\mathbf{v} = 0$:**
I set up the augmented matrix:

$$\begin{pmatrix} \frac{-1-\sqrt{5}}{2} & i \\ -i & \frac{1-\sqrt{5}}{2} \end{pmatrix}$$

I performed row reduction:
- Normalize row 1:

$$R1 \to 2 \cdot R1 \implies \begin{pmatrix} -1-\sqrt{5} & 2i \\ -i & \frac{1-\sqrt{5}}{2} \end{pmatrix}$$

- Eliminate $-i$ from row 2:

$$R2 \to R2 + i \cdot R1 \implies \begin{pmatrix} -1-\sqrt{5} & 2i \\ 0 & 0 \end{pmatrix}$$

Solving this system:

$$-1 - \sqrt{5} \cdot x + 2i \cdot y = 0$$

Choose $y = 1$:

$$x = \frac{2i}{-1-\sqrt{5}}$$

Thus, the eigenvector corresponding to λ_1 is:

$$\mathbf{v}_1 = \begin{pmatrix} \frac{2i}{-1-\sqrt{5}} \\ 1 \end{pmatrix}$$

For $\lambda_2 = \frac{5-\sqrt{5}}{2}$

1. **Compute $A - \lambda_2 I$:**

$$A - \lambda_2 I = \begin{pmatrix} 2 - \frac{5-\sqrt{5}}{2} & i \\ -i & 3 - \frac{5-\sqrt{5}}{2} \end{pmatrix}$$

Simplifying:

$$2 - \frac{5-\sqrt{5}}{2} = \frac{4-(5-\sqrt{5})}{2} = \frac{-1+\sqrt{5}}{2}$$

$$3 - \frac{5-\sqrt{5}}{2} = \frac{6-(5-\sqrt{5})}{2} = \frac{1+\sqrt{5}}{2}$$

So:

$$A - \lambda_2 I = \begin{pmatrix} \frac{-1+\sqrt{5}}{2} & i \\ -i & \frac{1+\sqrt{5}}{2} \end{pmatrix}$$

2. **Solve $(A - \lambda_2 I)\mathbf{v} = 0$:**
I set up the augmented matrix:

$$\begin{pmatrix} \frac{-1+\sqrt{5}}{2} & i \\ -i & \frac{1+\sqrt{5}}{2} \end{pmatrix}$$

I performed row reduction:
- Normalize row 1:

$$R1 \to 2 \cdot R1 \implies \begin{pmatrix} -1+\sqrt{5} & 2i \\ -i & \frac{1+\sqrt{5}}{2} \end{pmatrix}$$

- Eliminate $-i$ from row 2:

$$R2 \to R2 + i \cdot R1 \implies \begin{pmatrix} -1+\sqrt{5} & 2i \\ 0 & 0 \end{pmatrix}$$

Solving this system:

$$-1+\sqrt{5} \cdot x + 2i \cdot y = 0$$

Choose $y = 1$:

$$x = \frac{2i}{-1+\sqrt{5}}$$

Thus, the eigenvector corresponding to λ_2 is:

$$\mathbf{v}_2 = \begin{pmatrix} \frac{2i}{-1+\sqrt{5}} \\ 1 \end{pmatrix}$$

Verify Orthogonality

To verify orthogonality, I computed the dot product, which is 2... Where is my error here?

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \left(\frac{2i}{-1-\sqrt{5}}\right)\left(\frac{2i}{-1+\sqrt{5}}\right) + 1 \times 1$$

Calculating:

$$\left(\frac{2i}{-1-\sqrt{5}}\right)\left(\frac{2i}{-1+\sqrt{5}}\right) = \frac{4i^2}{(-1-\sqrt{5})(-1+\sqrt{5})}$$

Since $i^2 = -1$:

$$\frac{4i^2}{(-1-\sqrt{5})(-1+\sqrt{5})} = \frac{-4}{(-1)^2 - (\sqrt{5})^2}$$
$$= \frac{-4}{1-5} = \frac{-4}{-4} = 1$$

So:

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 1 + 1 = 2$$