

The role of acceleration and locality in the twin paradox

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Abstract

We study the role of acceleration in the twin paradox. From the coordinate transformation that relates an accelerated and an inertial observer we find that, from the point of view of the accelerated observer, the rate of the differential lapses of time depends not only on the relative velocity, but also on the product of the acceleration and the distance between the observers. However, this result does not have a direct operational interpretation because an observer at a certain position can measure only physical quantities that are defined at the same position. For local measurements, the asymmetry between the two observers can be attributed to the fact that noninertial coordinate systems, contrary to inertial coordinate systems, can be correctly interpreted only locally.

1 Introduction

According to the special theory of relativity, all motions with a constant velocity are relative. The twin paradox consists in the fact that a twin B that travels around and eventually meets his brother A is younger than his brother A . How the twin B knows that he is the one who is actually moving? An often answer, especially in the older literature, is: “He knows, because he accelerates and consequently feels an inertial force.” However, it has been stressed many times in the literature^(1–5) that acceleration is not an essential part of the twin paradox.

The most general explanation of the twin paradox is purely geometrical; the proper lengths of two trajectories in spacetime, which correspond to times

measured by the corresponding observers, do not need to be equal. This explanation is perfectly correct and is also the simplest one, because one does not need an explicit coordinate transformation that relates the two observers. Yet, such an explanation may not be completely satisfactory, because one may want to know where and when the different aging of the two observers occurs. Although, strictly speaking, this question is not really meaningful,⁽⁴⁾ we show in Sec. 2 that, in a certain sense, the different aging can be attributed to instants of time when one of the observers accelerates. We find that the accelerated observer “observes” that the rate of the differential lapses of time depends not only on the relative velocity, but also on the product of the acceleration and the distance between the observers. This result has already been obtained in Ref. 6, but our derivation is quite different, and, we believe, more elegant. However, we also emphasize that this result does not have a direct operational interpretation because an observer at a certain position can only measure physical quantities that are defined at the same position.

Another way of posing the twin paradox is to ask what, if not acceleration, is the source of asymmetry between the two relatively moving observers. When spacetime is curved⁽¹⁾ or has a nontrivial topology,^(3,7,8) one can obtain the twin paradox completely without acceleration. In these two cases it is clear that, owing to a nontrivial geometry or topology, there is no symmetry with respect to rotations of the velocity directions. However, a flat opened universe possesses such a symmetry, so the question of the source of asymmetry for such a case remains opened. We find in Sec. 3 that this asymmetry can be attributed to the fact that noninertial coordinate systems, contrary to inertial coordinate systems, can be correctly interpreted only locally. Recently, this fact has been used to resolve the Ehrenfest paradox,⁽⁹⁾ to give the correct interpretation of the Sagnac effect,⁽⁹⁾ and to show that the notion of radiation does not depend on acceleration of an observer.⁽¹⁰⁾ In this paper we explain how this fact helps in understanding the twin paradox.

2 The role of acceleration

Let S be the frame of an inertial observer and S' the frame of an observer that moves arbitrarily along the x -axis. S is a Lorentz frame, while S' is actually a coordinate system determined by the Fermi-Walker transport along the trajectory of the arbitrarily moving observer. (We give more comments on this in Sec. 3).

Let u be the velocity of the observer in S' as seen by an observer in S . Let us also assume that the observers in S and S' do not rotate (this makes the analysis simpler, but has no influence on the final results). The coordinate

transformation between these two frames is given by^(11,9) $y = y'$, $z = z'$, and

$$x = \gamma(t')x' + \int_0^{t'} \gamma(t')u(t')dt' , \quad (1)$$

$$t = \frac{1}{c^2}\gamma(t')u(t')x' + \int_0^{t'} \gamma(t')dt' , \quad (2)$$

where $\gamma(t') = 1/\sqrt{1 - u^2(t')/c^2}$. It was assumed in this transformation that the space origins coincide at $t = t' = 0$. The position of the observer in S' is $x' = 0$.

The transformation (1)-(2) is linear in x' . However, if $u(t')$ is not a constant, then this transformation is not linear in t' . Contrary to the case of constant u , this transformation cannot be simply inverted by putting $u \rightarrow -u$. This is why the inertial and the noninertial observers are not equivalent. Note, however, that acceleration does not appear explicitly in (1) and (2).

Let us now see how the clock in S' appears to the observer in S . Since the clock in S' is at $x' = 0$, from (2) we find

$$t = \int_0^{t'} \gamma(t')dt' , \quad (3)$$

$$\frac{\partial t}{\partial t'} = \gamma(t') . \quad (4)$$

Equations (3) and (4) express the fact that the observer in S sees that the clock in S' is slower than the clock in S and that $\partial t/\partial t'$ does not depend on the acceleration, but only on the instantaneous velocity.

Let us now see how the clock in S , not necessarily at $x = 0$, appears to the observer in S' . By eliminating x' from (1) and (2), we find

$$t = \int_0^{t'} \gamma(t')dt' + \frac{1}{c^2}u(t')[x - x_o(t')] , \quad (5)$$

where

$$x_o(t') = \int_0^{t'} \gamma(t')u(t')dt' = \int_0^{t(t')} u(t'(t))dt \quad (6)$$

is the position of the observer in S' as a function of time, $t(t')$ is given by (3), and $t'(t)$ is its inverse. It was allowed to use (3) and (4) in the second equality in (6) because, although u is expressed as a function of t' , it is, by definition, the velocity seen by the S -observer, i.e., $u = dx_o/dt$. From (5) and (6) we find

$$\frac{\partial t}{\partial t'} = \frac{1}{\gamma(t')} + \frac{1}{c^2} \frac{du(t')}{dt'} [x - x_o(t')] . \quad (7)$$

The fact that (7) and (4) are different is a consequence of the fact that the partial derivative in (7) is calculated with x held fixed, while the partial

derivative in (4) is calculated with x' held fixed. Comparing (7) with (4), we see that the first term in (7) corresponds to what we expect from the relativity of motion. However, the second term in (7) shows that acceleration has a direct influence on what the accelerated observer will observe. Note also that the influence of acceleration does not depend only on the acceleration itself, but also on the relative distance between the accelerated observer and the inertial clock. In particular, if the inertial and the noninertial clocks are at the same instantaneous position, then acceleration has no influence.

Now it seems that we understand what is the true origin of the different aging of the inertial and the noninertial clocks. For example, if the observer in S' moves with a constant velocity and then suddenly reverses the direction of motion, then, at this critical instant of time, it will appear to him that the time of the inertial clock at $x = 0$ instantaneously jumps forward. There is no such jump of the noninertial clock from the point of view of the inertial observer. This is the reason that, when the two observers finally meet, they have different age. A similar conclusion, although obtained in a completely different way, was drawn also in Ref. 13. However, as we show in the next section, this is not the end of the story.

3 The role of locality

The discussion of the preceding section seems to resolve the twin paradox. However, this discussion raises a new paradox. The right-hand side of (7) can be negative, which means that it may appear to the accelerated observer that an inertial clock lapses backward in time. This seems to be in contradiction with the principle of causality.

This paradox is an artefact of the tacit assumption that an observer receives information from a distant clock *instantaneously*. In that sense, equations (3), (4), (5), and (7) do not represent what the observers will really see, unless the clocks and the observers in S and S' are at the same instantaneous position. One could calculate what the observers would really see by assuming that the observers communicate with light signals, which would remove the causality paradox. However, we do not want to introduce a new entity, such as a light beam needed for communication, because such a complication could hide the real origin of the twin paradox. Instead, we insist on resolving the twin paradox using only the properties of the transformation (1)-(2).

An observer at a certain position can measure only the values of physical quantities at this same position. Therefore, equations (5) and (7) have a direct operational interpretation only for $x = x_o(t')$. Therefore, when the observer in S' compares his clock with a clock in S at the same instantaneous

position, he sees

$$t = \int_0^{t'} \gamma(t') dt' , \quad (8)$$

$$\frac{\partial t}{\partial t'} = \frac{1}{\gamma(t')} . \quad (9)$$

The apparent inconsistency of equations (8) and (9) is a new way of viewing the twin paradox. Equations (9) and (4) correspond to the relativity of motion; the two observers do not agree on which clock is faster. On the other hand, equations (8) and (3) correspond to the twin paradox; the two observers *do* agree that, at the same instant and the same position, the hand of the clock in S' points to a smaller number than the hand of the clock in S . But how is that possible?

Note first that the apparent inconsistency of (8) and (9) has a simple mathematical origin. One should not derive (9) directly from (8), but instead one needs *first* to calculate the derivative of (5) and *then* to put $x = x_o(t')$. However, (8) and (9) are correct equations even for a motion with a constant velocity. What is the source of asymmetry between S and S' ?

The asymmetry lies in another tacit assumption; the condition $x = x_o(t')$ corresponds to an experimental arrangement in which the moving clock in S' is compared each time with *another* clock in S , that which, at this instant, is at the same position as the clock in S' . In other words, there is precisely one clock in S' , while there are many clocks in S distributed along the x -axis.

One will say: “OK, but we can arrange our experiment such that there is only one clock in S , say at $x = 0$, while there are many clocks in S' distributed along the x' -axis.” However, now comes the essential point of this section. It was legitimate to use many clocks at different positions in S and compare the clock in S' each time with another clock in S , because all these clocks in S belong to the *same* frame of reference. On the other hand, if, at least for a short time, S' is not an inertial frame, then we cannot longer say that clocks at different constant positions x' belong to the same frame of reference. In other words, the proper coordinates related to the trajectory $x' = \text{constant}$ depend on this constant. Consequently, it is not legitimate to compare the inertial clock at $x = 0$ each time with another noninertial clock that does not move with respect to the noninertial clock at $x' = 0$ and interpret the result as something that tell us about the behavior of the clock at $x' = 0$. In the context of the twin paradox, a privileged role of inertial coordinate systems is not related to the fact that inertial observers do not feel an inertial force, but rather to the fact that inertial coordinate systems in flat spacetime have a clear *global* interpretation, while noninertial coordinate systems only have a clear *local* interpretation. This purely local interpretation of noninertial coordinate systems has already been explained in more detail in Refs. 9. and 10., but, for the sake of completeness, below we give a short resume of

the results of these papers.

Proper coordinates of an observer arbitrarily moving in arbitrary spacetime are determined by the Fermi-Walker transport ⁽¹²⁾ and are often referred to as Fermi coordinates.⁽¹⁵⁾ They are determined by the trajectory of the observer, as well as by the geometry of spacetime. It is convenient to define them such that the position of the observer is at the space origin of the Fermi coordinates. Even if there is no relative motion between two observers, they belong to different Fermi frames if they are not staying at the same position. In particular, the S -coordinates in (1)-(2) are the Fermi coordinates of an inertial observer in flat spacetime, while the S' -coordinates are the Fermi coordinates of an observer that moves arbitrarily (without rotation) along the x -axis with respect to the inertial observer. In general, the coordinate transformation that relates the Fermi coordinates of two different observers is a complicated transformation. However, the coordinate transformation that relates the Fermi coordinates of two inertial observers in flat spacetime that do not move relatively to each other is a simple translation of the space origin, which is a transformation that belongs to the class of restricted internal transformations,⁽⁹⁾ i.e., it is a transformation that does not mix space and time coordinates:

$$t' = f^0(t) , \quad x'^i = f^i(x^1, x^2, x^3) . \quad (10)$$

If the Fermi coordinates of two observers are related by such a transformation, then they can be regarded as belonging to the same *physical* frame of reference. This is why one can regard the clocks at different positions x in (5) as belonging to the same frame of reference.

Now we also have a better understanding of the causality “paradox” mentioned at the beginning of this section; it is merely a consequence of the fact that the coordinate t' can be interpreted as a physical time only at $x' = 0$.

At the end, let us also mention some measurable effects related to the local interpretation of noninertial frames. Assume that the S' -coordinates in (1)-(2) refer to a uniformly accelerated observer at $x' = 0$. Assume also that another observer has such a trajectory that his position is given by $x' = \text{constant} \neq 0$. Then this second observer also accelerates uniformly, but with a *different* acceleration.⁽¹⁴⁾ If, on the other hand, two observers at different positions move in the same direction with equal accelerations and initial velocities, then, as seen by these observers, the distance between them changes with time, which also leads to a variant of the twin paradox.⁽²⁾

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