† G. A Sharper Cayley Theorem

If H is a subgroup of a group G, let X designate the set of all the left cosets of H in G. For each element $a \in G$, define $\rho_a : X \to X$ as follows:

$$\rho_a(xH) = (ax)H$$

1 Prove that each ρ_a is a permutation of X.

2 Prove that $h : G \to S_X$ defined by $h(a) = \rho_a$ is a homomorphism.

\# 3 Prove that the set $\{a \in H : xax^{-1} \in H \text{ for every } x \in G\}$, that is, the set of all the elements of H whose conjugates are all in H, is the kernel of h.

4 Prove that if H contains no normal subgroup of G except $\{e\}$, then G is isomorphic to a subgroup of S_X.