PHYS-5311: Homework 3

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Contents

Chapter 1

Homework 1

1.1 Problem 1

1. A point particle moves in space under the influence of a force derivable from a generalized potential of the form:

$U(\mathbf{r}, \mathbf{v}) = V(r) + \boldsymbol{\sigma} \bullet \mathbf{L}$

where $\mathbf{r}$ is a radius vector from a fixed origin, $\mathbf{L}$ is the angular momentum about that point, and $\boldsymbol{\sigma}$ is a fixed vector in space.

a) Find the components of the generalized force on the particle in both Cartesian and spherical polar coordinates according to $Q_j = -\partial U/\partial q_j + d/dt[\partial U/\partial(dq_j/dt)]$.

b) Show that the components in the two coordinate systems are related according to $Q_j = \Sigma_i \mathbf{F}_i \bullet (\partial \mathbf{r}_i/\partial q_j)$.

c) Obtain the equations of motion in spherical coordinates.

1.1.1 Part a)

Number of objects	N	1
Number of Dimensions	D	3
Number of translation D.O.F	$N_{\text{trans}} = ND$	3
Object dimensionality	C	0
Number of rotational D.O.F	$N_{\text{rot}} = \frac{1}{2}D(D-1) - \frac{1}{2}(D-C)(D-C-1)$	0
Number of constraints	M	0
Number of D.O.F	$f = N_{\text{trans}} + N_{\text{rot}} - M$	3

- Cartesian Coordinates

  Since there are 3 D.O.F and no constraints, I will choose X, Y, Z as the G.Cs. First, focus on getting the potential. Let

$$\mathbf{R} = \left(\hat{\mathbf{x}}X + \hat{\mathbf{y}}Y + \hat{\mathbf{z}}Z\right)$$

  be the position of the particle relative to the origin $\mathcal{O}$.Let

$$\sigma = \left(\hat{\mathbf{x}}\sigma_x + \hat{\mathbf{y}}\sigma_y + \hat{\mathbf{z}}\sigma_z\right)$$

.

Now, the momentum of the particle is just.

$$\mathbf{P} = m\Big(\hat{\mathbf{x}}\dot{X} + \hat{\mathbf{y}}\dot{Y} + \hat{\mathbf{z}}\dot{Z}\Big)$$

The angular momentum is about the origin; thus,

$$L_i = \varepsilon_{ijk} X_j p_k$$

This is the ith component of the angular momentum given X_j is the jth component of the position of the particle relative to the origin and p_k is the kth component of the linear momentum of the particle.

$$L_i = m\varepsilon_{ijk} X_j \dot{X}_k$$

$$L_1 = m\Big[X_2\dot{X}_3 - X_3\dot{X}_2\Big]$$

We can permute indices in order to get both L_2 and L_3.

$$L_2 = m\Big[X_3\dot{X}_1 - X_1\dot{X}_3\Big]$$

$$L_3 = m\Big[X_1\dot{X}_2 - X_2\dot{X}_1\Big]$$

So,

$$\begin{cases} L_x = m\Big[Y\dot{Z} - Z\dot{Y}\Big] \\ \\ L_y = m\Big[Z\dot{X} - X\dot{Z}\Big] \\ \\ L_z = m\Big[X\dot{Y} - Y\dot{X}\Big] \end{cases}$$

Now,

$$\boldsymbol{\sigma} \cdot \mathbf{L} = \sigma_i L_i = \sigma_i \varepsilon_{ijk} X_j p_k$$

Thus, the potential becomes

$$U = V(X,Y,Z)) + m\Big(\sigma_x\Big[Y\dot{Z} - Z\dot{Y}\Big] + \sigma_y\Big[Z\dot{X} - X\dot{Z}\Big] + \sigma_z\Big[X\dot{Y} - Y\dot{X}\Big]\Big)$$

$$U = V(X,Y,Z) + m\Big([\sigma_y Z - \sigma_z Y]\dot{X} + [\sigma_z X - \sigma_x Z]\dot{Y} + [\sigma_x Y - \sigma_y X]\dot{Z}\Big)$$

Now using

$$Q_i = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\Big(\frac{\partial U}{\partial \dot{q}_j}\Big)$$

$$Q_X = -\frac{\partial V}{\partial X} + m\Big[\sigma_z \dot{Y} - \sigma_y \dot{Z}\Big] + m\frac{d}{dt}\Big(\sigma_y Z - \sigma_z Y\Big)$$

Assuming $\boldsymbol{\sigma}$ is a constant vector, since it is fixed in space

$$Q_X = -\frac{\partial V}{\partial X} - m\Big[\sigma_z \dot{Y} - \sigma_y \dot{Z}\Big] + m\Big[\sigma_y \dot{Z} - \sigma_z \dot{Y}\Big]$$

Permuting indices

$$Q_Y = -\frac{\partial V}{\partial Y} - m\Big[\sigma_x \dot{Z} - \sigma_z \dot{X}\Big] + m\Big[\sigma_z \dot{X} - \sigma_x \dot{Z}\Big]$$
$$Q_Z = -\frac{\partial V}{\partial Z} - m\Big[\sigma_y \dot{X} - \sigma_x \dot{Y}\Big] + m\Big[\sigma_x \dot{Y} - \sigma_y \dot{X}\Big]$$

Equivalently

$$Q_X = -\frac{\partial V}{\partial X} + 2m[\boldsymbol{\sigma} \times \dot{\mathbf{R}}]_X$$

Thus, in cartesian coordinates

$$\boxed{Q_i = \partial_i V + 2m\epsilon_{ipq}\sigma_p \dot{X}_q}$$

In Vector form

$$\boxed{\mathbf{Q} = -\nabla V + 2m\boldsymbol{\sigma} \times \dot{\mathbf{R}}}$$

- Spherical coordinates

  This time choose R, Φ, Θ as the G.Cs and again focus on getting the potential

$$\mathbf{R} = \hat{\mathbf{r}}(\Phi, \Theta)R$$

And, choose

$$\boldsymbol{\sigma} = \hat{\mathbf{z}}\sigma$$

From part a)

$$\mathbf{Q} = -\nabla V + 2m\boldsymbol{\sigma} \times \dot{\mathbf{R}}$$

Since $V(R)$ is a function of only R, then from $A.9$ we know that

$$-\nabla V = -\hat{\mathbf{R}}\frac{dV}{dR}$$

Now,

$$m\boldsymbol{\sigma}\times\dot{\mathbf{R}} = m\sigma\Big[\dot{R}\cos(\Phi)\sin(\Theta)-R\sin(\Phi)\dot{\Phi}\sin(\Theta)+R\cos(\Phi)\cos(\Theta)\dot{\Theta}\Big]\underbrace{(\hat{\mathbf{z}}\times\hat{\mathbf{x}})}_{\hat{\mathbf{y}}}$$

$$+m\sigma\Big[\dot{R}\sin(\Phi)\sin(\Theta)+R\cos(\Phi)\dot{\Phi}\sin(\Theta)+R\sin(\Phi)\cos(\Theta)\dot{\Theta}\Big]\underbrace{(\hat{\mathbf{z}}\times\hat{\mathbf{y}})}_{-\hat{\mathbf{x}}}$$

From $A.11$ and $A.10$

$$m\boldsymbol{\sigma}\times\dot{\mathbf{R}} = m\sigma\Big[\dot{R}\cos(\Phi)\sin(\Theta)-R\sin(\Phi)\dot{\Phi}\sin(\Theta)+R\cos(\Phi)\cos(\Theta)\dot{\Theta}\Big]$$

$$\Big[\hat{\mathbf{R}}\sin(\Phi)\sin(\Theta)+\hat{\boldsymbol{\Phi}}\cos(\Phi)+\hat{\boldsymbol{\Theta}}\sin(\Phi)\cos(\Theta)\Big]$$

$$+m\sigma\Big[-\dot{R}\sin(\Phi)\sin(\Theta)-R\cos(\Phi)\dot{\Phi}\sin(\Theta)-R\sin(\Phi)\cos(\Theta)\dot{\Theta}\Big]$$

$$\Big[\hat{\mathbf{R}}\cos(\Phi)\sin(\Theta)-\hat{\boldsymbol{\Phi}}\sin(\Phi)+\hat{\boldsymbol{\Theta}}\cos(\Phi)\cos(\Theta)\Big]$$

$$m\boldsymbol{\sigma}\times\dot{\mathbf{R}} = m\sigma\Big[-\hat{\mathbf{R}}R\sin^2(\Theta)\dot{\Phi}+\hat{\boldsymbol{\Phi}}\Big(\dot{R}\sin^2(\Theta)+R\sin(\Theta)\cos(\Theta)\dot{\Theta}\Big)-\hat{\boldsymbol{\Theta}}R\dot{\Phi}\sin(\Theta)\cos(\Theta)\Big]$$

So, we now have

$$Q_R = -\frac{dV}{dR} - 2\sigma mR\sin^2(\Theta)\dot{\Phi}$$

$$Q_\Phi = 2\sigma m\dot{R}\sin(\Theta) + 2\sigma mR\cos(\Theta)\dot{\Theta}$$

$$Q_\Theta = -2\sigma mR\dot{\Phi}\sin(\Theta)\cos(\Theta)$$

1.1.2 Part b)

We want to show

$$Q_j = \sum_i \mathbf{F}_i\cdot\frac{\partial \mathbf{r}_i}{\partial q_j} \tag{1.1}$$

is true.

Compare to the expression from cartesian coordinates with $\sigma_x = \sigma_y = 0$ and $\sigma_z = \sigma$.

Cartesian	Spherical
$Q_X = -\frac{\partial V}{\partial X} - 2\sigma m\dot{Y}$	$Q_R = -\frac{dV}{dR} - 2\sigma mR\sin^2(\Theta)\dot{\Phi}$
$Q_Y = -\frac{\partial V}{\partial Y} + 2\sigma m\dot{X}$	$Q_\Phi = 2\sigma m\dot{R}\sin(\Theta) + 2\sigma mR\cos(\Theta)\dot{\Theta}$
$Q_Z = -\frac{\partial V}{\partial Z}$	$Q_\Theta = -2\sigma mR\dot{\Phi}\sin(\Theta)\cos(\Theta)$

Starting with Q_R

$$Q_R = \Big(-\frac{\partial V}{\partial X} - 2\sigma m\dot{Y}\Big)\frac{\partial X}{\partial R} + \Big(-\frac{\partial V}{\partial Y} + 2\sigma m\dot{X}\Big)\frac{\partial Y}{\partial R} + \Big(-\frac{\partial V}{\partial Z}\Big)\frac{\partial Z}{\partial R}$$

$$Q_R = -\frac{\partial V}{\partial X}\frac{\partial X}{\partial R} - 2\sigma m\dot{Y}\frac{\partial X}{\partial R} - \frac{\partial V}{\partial Y}\frac{\partial Y}{\partial R} + 2\sigma m\dot{X}\frac{\partial Y}{\partial R} - \frac{\partial V}{\partial Z}\frac{\partial Z}{\partial R}$$

$$Q_R = -\frac{dV}{dR} - 2\sigma m\dot{Y}\frac{\partial X}{\partial R} + 2\sigma m\dot{X}\frac{\partial Y}{\partial R}$$

$$Q_R = -\frac{dV}{dR} - 2\sigma m\Big(\dot{R}\sin(\Phi)\sin(\Theta) + R\cos(\Phi)\sin(\Theta)\dot{\Phi} + R\sin(\Phi)\cos(\Theta)\dot{\Theta}\Big)\frac{\partial X}{\partial R}$$
$$+2\sigma m\Big(\dot{R}\cos(\Phi)\sin(\Theta) - R\sin(\Phi)\sin(\Theta)\dot{\Phi} + R\cos(\Phi)\cos(\Theta)\dot{\Theta}\Big)\frac{\partial Y}{\partial R}$$

$$Q_R = -\frac{dV}{dR} - 2\sigma m\Big(\dot{R}\sin(\Phi)\sin(\Theta) + R\cos(\Phi)\sin(\Theta)\dot{\Phi} + R\sin(\Phi)\cos(\Theta)\dot{\Theta}\Big)\Big(\cos(\Phi)\sin(\Theta)\Big)$$
$$+2\sigma m\Big(\dot{R}\cos(\Phi)\sin(\Theta) - R\sin(\Phi)\sin(\Theta)\dot{\Phi} + R\cos(\Phi)\cos(\Theta)\dot{\Theta}\Big)\Big(\sin(\Phi)\sin(\Theta)\Big)$$

$$Q_R = -\frac{dV}{dR} - 2\sigma m R\sin^2(\Theta)\dot{\Phi} \tag{1.2}$$

Continuing with Q_Φ

$$Q_\Phi = \Big(-\frac{\partial V}{\partial X} - 2\sigma m\dot{Y}\Big)\frac{\partial X}{\partial \Phi} + \Big(-\frac{\partial V}{\partial Y} + 2\sigma m\dot{X}\Big)\frac{\partial Y}{\partial \Phi} + \Big(-\frac{\partial V}{\partial Z}\Big)\frac{\partial Z}{\partial \Phi}$$

$$Q_\Phi = -\frac{\partial V}{\partial X}\frac{\partial X}{\partial \Phi} - 2\sigma m\dot{Y}\frac{\partial X}{\partial \Phi} - \frac{\partial V}{\partial Y}\frac{\partial Y}{\partial \Phi} + 2\sigma m\dot{X}\frac{\partial Y}{\partial \Phi} - \frac{\partial V}{\partial Z}\frac{\partial Z}{\partial \Phi}$$

$$Q_\Phi = -\frac{dV}{d\Phi} - 2\sigma m\dot{Y}\frac{\partial X}{\partial \Phi} + 2\sigma m\dot{X}\frac{\partial Y}{\partial \Phi}$$

However, since $V(R)$ is independent of Φ, then

$$Q_\Phi = -2\sigma m\dot{Y}\frac{\partial X}{\partial \Phi} + 2\sigma m\dot{X}\frac{\partial Y}{\partial \Phi}$$

Now

$$Q_\Phi = -2\sigma m\Big(\dot{R}\sin(\Phi)\sin(\Theta) + R\cos(\Phi)\sin(\Theta)\dot{\Phi} + R\sin(\Phi)\cos(\Theta)\dot{\Theta}\Big)\frac{\partial X}{\partial \Phi}$$
$$+2\sigma m\Big(\dot{R}\cos(\Phi)\sin(\Theta) - R\sin(\Phi)\sin(\Theta)\dot{\Phi} + R\cos(\Phi)\cos(\Theta)\dot{\Theta}\Big)\frac{\partial Y}{\partial \Phi}$$

$$Q_\Phi = -2\sigma m\Big(\dot{R}\sin(\Phi)\sin(\Theta) + R\cos(\Phi)\sin(\Theta)\dot{\Phi} + R\sin(\Phi)\cos(\Theta)\dot{\Theta}\Big)\Big(-R\sin(\Phi)\sin(\Theta)\Big)$$
$$+2\sigma m\Big(\dot{R}\cos(\Phi)\sin(\Theta) - R\sin(\Phi)\sin(\Theta)\dot{\Phi} + R\cos(\Phi)\cos(\Theta)\dot{\Theta}\Big)\Big(R\cos(\Phi)\sin(\Theta)\Big)$$

(1.3) $$Q_\Phi = 2\sigma m\dot{R}\sin^2(\Theta) + 2\sigma mR\sin(\Theta)\cos(\Theta)\dot{\Theta}$$

Finally, Q_Θ

$$Q_\Theta = \Big(-\frac{\partial V}{\partial X} - 2\sigma m\dot{Y}\Big)\frac{\partial X}{\partial \Theta} + \Big(-\frac{\partial V}{\partial Y} + 2\sigma m\dot{X}\Big)\frac{\partial Y}{\partial \Theta} + \Big(-\frac{\partial V}{\partial Z}\Big)\frac{\partial Z}{\partial \Theta}$$

$$Q_\Theta = -\frac{\partial V}{\partial X}\frac{\partial X}{\partial \Theta} - 2\sigma m\dot{Y}\frac{\partial X}{\partial \Theta} - \frac{\partial V}{\partial Y}\frac{\partial Y}{\partial \Theta} + 2\sigma m\dot{X}\frac{\partial Y}{\partial \Theta} - \frac{\partial V}{\partial Z}\frac{\partial Z}{\partial \Theta}$$

$$Q_\Theta = -\frac{dV}{d\Theta} - 2\sigma m\dot{Y}\frac{\partial X}{\partial \Theta} + 2\sigma m\dot{X}\frac{\partial Y}{\partial \Theta}$$

However, $V(R)$ does not depend on Θ; thus,

$$Q_\Theta = -2\sigma m\dot{Y}\frac{\partial X}{\partial \Theta} + 2\sigma m\dot{X}\frac{\partial Y}{\partial \Theta}$$

Now

$$Q_\Theta = -2\sigma m\Big(\dot{R}\sin(\Phi)\sin(\Theta) + R\cos(\Phi)\sin(\Theta)\dot{\Phi} + R\sin(\Phi)\cos(\Theta)\dot{\Theta}\Big)\frac{\partial X}{\partial \Theta}$$
$$+2\sigma m\Big(\dot{R}\cos(\Phi)\sin(\Theta) - R\sin(\Phi)\sin(\Theta)\dot{\Phi} + R\cos(\Phi)\cos(\Theta)\dot{\Theta}\Big)\frac{\partial Y}{\partial \Theta}$$

$$Q_\Theta = -2\sigma m\Big(\dot{R}\sin(\Phi)\sin(\Theta) + R\cos(\Phi)\sin(\Theta)\dot{\Phi} + R\sin(\Phi)\cos(\Theta)\dot{\Theta}\Big)\Big(R\cos(\Phi)\cos(\Theta)\Big)$$
$$+2\sigma m\Big(\dot{R}\cos(\Phi)\sin(\Theta) - R\sin(\Phi)\sin(\Theta)\dot{\Phi} + R\cos(\Phi)\cos(\Theta)\dot{\Theta}\Big)\Big(R\sin(\Phi)\cos(\Theta)\Big)$$

(1.4) $$Q_\Theta = -2\sigma mR\dot{\Phi}\sin(\Theta)\cos(\Theta)$$

We can see that 1.1 holds from 1.2, 1.3, and 1.4

1.1.3 Part c)

Next, the kinetic energy.

$$T = T_{\text{trans}} + T_{\text{rot}}$$

From $A.4$

$$T_{\text{trans}} = \frac{1}{2}mv^2 = \frac{1}{2}m\Big(\dot{R}^2 + R^2\sin^2(\Theta)\dot{\phi}^2 + R^2\dot{\Theta}^2\Big)$$

Since there are no rotational D.O.F for the particle, then

$$T_{\text{rot}} = 0$$

Thus,

$$\mathcal{L} = \frac{1}{2}m\Big(\dot{R}^2 + R^2\sin^2(\Theta)\dot{\phi}^2 + R^2\dot{\Theta}^2\Big)$$

$$-V(R) - m\Big([\sigma_y Z - \sigma_z Y]\dot{X} + [\sigma_z X - \sigma_x Z]\dot{Y} + [\sigma_x Y - \sigma_y X]\dot{Z}\Big)$$

More compactly, since $\sigma_x = \sigma_y = 0$ and $\sigma_z = \sigma$.

$$\mathcal{L} = \frac{1}{2}m\Big(\dot{R}^2 + R^2\sin^2(\Theta)\dot{\Phi}^2 + R^2\dot{\Theta}^2\Big) - V(R) - \sigma m\Big(X\dot{Y} - Y\dot{X}\Big)$$

$$\mathcal{L} = \frac{1}{2}m\Big(\dot{R}^2 + R^2\sin^2(\Theta)\dot{\Phi}^2 + R^2\dot{\Theta}^2\Big) - V(R)$$

$$-\sigma m\Big(R\cos(\Phi)\sin(\Theta)\Big[\dot{R}\sin(\Phi)\sin(\Theta) + R\cos(\Phi)\sin(\Theta)\dot{\Phi} + R\sin(\Phi)\cos(\Theta)\dot{\Theta}\Big]$$

$$-R\sin(\Phi)\sin(\Theta)\Big[\dot{R}\cos(\Phi)\sin(\Theta) - R\sin(\Phi)\sin(\Theta)\dot{\Phi} + R\cos(\Phi)\cos(\Theta)\dot{\Theta}\Big]\Big)$$

$$\mathcal{L} = \frac{1}{2}m\Big(\dot{R}^2 + R^2\sin^2(\Theta)\dot{\Phi}^2 + R^2\dot{\Theta}^2\Big) - V(R)$$

$$-m\sigma\Big(R\dot{R}\sin(\Phi)\cos(\Phi)\sin^2(\Theta) + R^2\cos^2(\Phi)\sin^2(\Theta)\dot{\Phi} + R^2\cos(\Phi)\sin(\Phi)\cos(\Theta)\sin(\Theta)\dot{\Theta}$$

$$-R\dot{R}\sin(\Phi)\cos(\Phi)\sin^2(\Theta) + R^2\sin^2(\Phi)\sin^2(\Theta)\dot{\Phi} - R^2\sin(\Phi)\cos(\Phi)\sin(\Theta)\cos(\Theta)\dot{\Theta}\Big)$$

$$\mathcal{L} = \frac{1}{2}m\Big(\dot{R}^2 + R^2\sin^2(\Theta)\dot{\Phi}^2 + R^2\dot{\Theta}^2\Big) - V(R) - \sigma m R^2\sin^2(\Theta)\dot{\Phi}$$

$$\frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{q}_j}\Big) - \frac{\partial\mathcal{L}}{\partial q_j} = 0 \tag{1.5}$$

For R:

$$\frac{\partial\mathcal{L}}{\partial\dot{R}} = m\dot{R}$$

$$\frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{R}}\Big) = m\ddot{R}$$

$$\frac{\partial\mathcal{L}}{\partial R} = mR\sin^2(\Theta)\dot{\Phi}^2 + mR\dot{\Theta}^2 - \frac{dV}{dR} - 2\sigma mR\sin^2(\Theta)\dot{\Phi}$$

$$\boxed{m\ddot{R} - mR\sin^2(\Theta)\dot{\Phi}^2 - mR\dot{\Theta}^2 = -\frac{dV}{dR} - 2\sigma mR\sin^2(\Theta)\dot{\Phi}} \tag{1.6}$$

For Φ

$$\frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = mR^2 \sin^2(\Theta)\dot{\Phi} - \sigma m R^2 \sin^2(\Theta)$$

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{\Phi}}\Big) = 2mR\dot{R}\sin^2(\Theta)\dot{\Phi} + 2mR^2\sin(\Theta)\cos(\Theta)\dot{\Theta}\dot{\Phi} + mR^2\sin^2(\Theta)\ddot{\Phi}$$

$$-2\sigma mR\dot{R}\sin^2(\Theta) - 2\sigma mR^2\sin(\Theta)\cos(\Theta)\dot{\Theta}$$

$$\frac{\partial \mathcal{L}}{\partial \Phi} = 0$$

So,

$$\begin{aligned} 2m\dot{R}\sin^2(\Theta)\dot{\Phi} + 2mR\sin(\Theta)\cos(\Theta)\dot{\Theta}\dot{\Phi} + mR\sin^2(\Theta)\ddot{\Phi} = \\ 2\sigma m\dot{R}\sin^2(\Theta) + 2\sigma mR\sin(\Theta)\cos(\Theta)\dot{\Theta} \end{aligned} \tag{1.7}$$

For Θ

$$\frac{\partial \mathcal{L}}{\partial \dot{\Theta}} = mR^2\dot{\Theta}$$

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{\Theta}}\Big) = 2mR\dot{R}\dot{\Theta} + mR^2\ddot{\Theta}$$

$$\frac{\partial \mathcal{L}}{\partial \Theta} = mR^2\sin(\Theta)\cos(\Theta)\dot{\Phi}^2 - 2\sigma mR^2\sin(\Theta)\cos(\Theta)\dot{\Phi}$$

$$2m\dot{R}\dot{\Theta} + mR\ddot{\Theta} - mR\sin(\Theta)\cos(\Theta)\dot{\Phi}^2 = -2\sigma mR\sin(\Theta)\cos(\Theta)\dot{\Phi} \tag{1.8}$$

1.2 Problem 2

2. Two particles of mass *m* are joined by a rigid massless rod of length *l*, the center of which is constrained to move on a circle of radius *a*. How many degrees of freedom does the system have if the particles can move in three dimensions? Express the kinetic energy in generalized coordinates.

First consider that there are two particles each in 3 dimensions. Since the particles are attached by a rod, this is one constraint; however, there is a second constraint since the center of the rod is constrained to move on a circle of radius a. Furthermore, the final constraint is that the circle is confined to the x-y plane. So, there are three constraints.

Number of objects	N	2
Number of Dimensions	D	3
Number of translation D.O.F	$N_{\text{trans}} = ND$	6
Object dimensionality	C	0
Number of rotational D.O.F	$N_{\text{rot}} = \frac{1}{2}D(D-1) - \frac{1}{2}(D-C)(D-C-1)$	0
Number of constraints	M	3
Number of D.O.F	$f = N_{\text{trans}} + N_{\text{rot}} - M$	3

Alternatively, consider a rod (1 dimensional object) whose center is constrained to move on a circle of radius a.Furthermore, the circle is confined to the x-y plane. This rod sits in 3 dimensions and there are two constraints.

Number of objects	N	1
Number of Dimensions	D	3
Number of translation D.O.F	$N_{\text{trans}} = ND$	3
Object dimensionality	C	1
Number of rotational D.O.F	$N_{\text{rot}} = \frac{1}{2}D(D-1) - \frac{1}{2}(D-C)(D-C-1)$	2
Number of constraints	M	2
Number of D.O.F	$f = N_{\text{trans}} + N_{\text{rot}} - M$	3

Let the center of the circle of radius a be the origin. Then choose ψ, θ, ϕ be the G.Cs: ψ is the angular displacement of the center of mass around the circle of radius a. Picturing a sphere with the origin at the center of mass of rod and of radius $\frac{l}{2}$, then θ is the latitudinal angle and ϕ is the longitudinal angle just like in spherical coordinates.

$$T = T_{\text{trans}} + T_{\text{rot}}$$

From $A.4$ we know what the velocity on the surface of a sphere with a constant radius of $\frac{l}{2}$ is; thus, for the kinetic energy we have:

$$T = \frac{1}{2}\mu a^2\dot{\psi}^2 + \frac{1}{2}(2m\frac{l^2}{4})\Big(\sin^2(\theta)\dot{\phi}^2 + \dot{\theta}^2\Big) \tag{1.9}$$

Where

$$\mu = \frac{m^2}{m+m} = \frac{m}{2} \tag{1.10}$$

So,

$$\boxed{T = \frac{m}{4}a^2\dot{\psi}^2 + \frac{m}{4}l^2\Big(\sin^2(\theta)\dot{\phi} + \dot{\theta}^2\Big)} \tag{1.11}$$

1.3 Problem 3

3. A particle of mass *m* moves in one dimension such that it has the Lagrangian:

L = m²(dx/dt)⁴/12 + m(dx/dt)²V(x) – V²(x)

where *V(x)* is some differentiable function of *x*. Find the equation of motion for *x(t)* and describe the physical nature of this system on the basis of this equation.

Using the E.L equation

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{x}}\Big) - \frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial \dot{x}} = \frac{m^2\dot{x}^3}{3} + 2m\dot{x}V(x)$$

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{x}}\Big) = m^2\dot{x}^2\ddot{x} + 2m\ddot{x}V(x) + 2m\dot{x}^2\frac{dV}{dx}$$

$$\frac{\partial L}{\partial x} = m\dot{x}^2\frac{dV}{dx} - 2V(x)\frac{dV}{dx}$$

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{x}}\Big) - \frac{\partial L}{\partial x} = m^2\dot{x}^2\ddot{x} + 2m\ddot{x}V(x) + 2m\dot{x}^2\frac{dV}{dx} - m\dot{x}^2\frac{dV}{dx} + 2V(x)\frac{dV}{dx}$$

$$= 2m\ddot{x}\Big(\frac{1}{2}m\dot{x}^2 + V(x)\Big) + 2\Big(\frac{1}{2}m\dot{x}^2 + V(x)\Big)\frac{dV}{dx}$$

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{x}}\Big) - \frac{\partial L}{\partial x} = 2\Big(m\ddot{x} + \frac{dV}{dx}\Big)\Big(\frac{1}{2}m\dot{x}^2 + V(x)\Big) = 0$$

So we can have

$$m\ddot{x} + \frac{dV}{dx} = 0 \tag{1.12}$$

or (disjunction)

$$\frac{1}{2}m\dot{x}^2 + V(x) = 0 \tag{1.13}$$

Choose equation 1.12 and set $\frac{1}{2}m\dot{x}^2 + V(x) = E$ where E is some constant; thus, we have a particle in an inertial frame. Equation 1.12 is just a statement of newton's second law with no external forces acting on the particle and

$$\frac{1}{2}m\dot{x}^2 + V(x) = E \tag{1.14}$$

is just the statement of conservation of energy.

Chapter 2

Homework 2

2.1 Problem 1

1. A point mass is constrained to move on a massless hoop of radius a in a vertical plane that rotates about the vertical direction with constant speed ω. Obtain the Lagrange equations of motion assuming that gravity is the only external force acting on the point mass. Show that if ω is greater than a critical value ω_0, there can be a solution in which the particle remains stationary at a point other than the bottom, but that if $\omega < \omega_0$, the only stationary point is at the bottom of the hoop. What is the value of ω_0?

We have a particle of mass m which is constrained to move on a hoop of radius a; further more, the angular velocity of the hoop is a constant ω. So, there are two constraints on a point particle

Number of objects	N	1
Number of Dimensions	D	3
Number of translation D.O.F	$N_{\text{trans}} = ND$	3
Object dimensionality	C	0
Number of rotational D.O.F	$N_{\text{rot}} = \frac{1}{2}D(D-1) - \frac{1}{2}(D-C)(D-C-1)$	0
Number of constraints	M	2
Number of D.O.F	$f = N_{\text{trans}} + N_{\text{rot}} - M$	1

Suppose that the pendulum is attached at the origin such that at rest, the particle rests at the bottom of the hoop. The most appropriate G.C for this problem is the polar angle from the $-z$ axis, θ where $0 \leq \theta \leq \pi$

The kinetic energy is

$$T = \frac{1}{2}m\left[a^2 \sin^2(\theta)\omega^2 + a^2\dot{\theta}^2\right] \tag{2.1}$$

The potential energy is

$$U = -mga\cos(\theta) \tag{2.2}$$

Thus, the lagrangian of the system is simply

$$\mathcal{L} = \frac{1}{2}m\Big[a^2\sin^2(\theta)\omega^2 + a^2\dot{\theta}^2\Big] + mga\cos(\theta) \tag{2.3}$$

Now using the Euler-Lagrange Equation for θ

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = ma^2\dot{\theta} \tag{2.4}$$

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\Big) = ma^2\ddot{\theta} \tag{2.5}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = ma^2\sin(\theta)\cos(\theta)\omega^2 - mga\sin(\theta) \tag{2.6}$$

Thus, the equation of motion for θ is just

$$\boxed{ma^2\ddot{\theta} - ma^2\sin(\theta)\cos(\theta)\omega^2 + mga\sin(\theta) = 0} \tag{2.7}$$

From (2.7)

$$\ddot{\theta} = \sin(\theta)\cos(\theta)\omega^2 - \frac{g}{a}\sin(\theta) \tag{2.8}$$

We seek a $\theta(t_c) = \theta_c$ which makes $\ddot{\theta}\Big|_{t_t c} = 0$ other than the trivial solution of $\theta_c = \pi$. Since by hypothesis $\theta_c \neq \pi$, then $\sin(\theta_c) \neq 0$.

At time t_c (2.8) becomes

$$\ddot{\theta}\Big|_{t=t_c} = \sin(\theta_c)\cos(\theta_c)\omega^2 - \frac{g}{a}\sin(\theta_c) \tag{2.9}$$

Suppose furthermore that $\ddot{\theta}\Big|_{t=t_c} = 0$, for $\omega = K\omega_0$ where $K > 0$ then

$$\sin(\theta_c)\cos(\theta_c)(K\omega_0)^2 - \frac{g}{a}\sin(\theta_c) = 0$$

$$\cos(\theta_c)K^2\omega_0^2 - \frac{g}{a} = 0$$

$$\cos(\theta_c) = \frac{g}{K^2\omega_0^2 a} \tag{2.10}$$

Let

$$\boxed{\omega_0 = \sqrt{\frac{g}{a}}} \tag{2.11}$$

Now (2.10) becomes

$$\cos(\theta_c) = \frac{1}{K^2} \tag{2.12}$$

If $K < 1$, then (2.12) has no solutions; thus, violating the hypothesis that a solution other than $\theta_c = \pi$ for $\ddot{\theta}\Big|_{t=t_c} = 0$ exists; however, if $K > 1$, then (2.12) has solutions and the exact value of θ_c depends on the K.

$$\theta_c = \cos^{-1}\left(\frac{1}{K^2}\right)$$

2.2 Problem 2

2. A particle of mass *m* is suspended by a massless string of length *L*. It hangs, without initial motion, in a gravitational field of strength *g*. It is struck by an impulsive horizontal blow, which introduces an angular velocity *ω*. If *ω* is sufficiently small, it is obvious that the mass moves as a simple pendulum. If *ω* is sufficiently large, the mass will rotate about the support. Use a Lagrange multiplier to determine the conditions under which the string becomes slack at some point in the motion.

Once the particle of mass m is imparted with angular velocity ω

Number of objects	N	1
Number of Dimensions	D	2
Number of translation D.O.F	$N_{\text{trans}} = ND$	2
Object dimensionality	C	0
Number of rotational D.O.F	$N_{\text{rot}} = \frac{1}{2}D(D-1) - \frac{1}{2}(D-C)(D-C-1)$	0
Number of constraints	M	1
Number of D.O.F	$f = N_{\text{trans}} + N_{\text{rot}} - M$	1

Let the particle be attached at the origin. Choose polar coordinates r and θ as the 2 generalized coordinates for the particle.

Now from (A.4) with $\dot{\phi} = 0$

$$T = \frac{1}{2}m\Big(\dot{r}^2 + r^2\dot{\theta}^2\Big) \tag{2.13}$$

$$U = -mgr\cos(\theta) \tag{2.14}$$

And the holonomic constraint is

$$f(r) = r - l = 0 \tag{2.15}$$

The forces of constraint Q_j are given by

$$Q_j = \sum_\alpha \lambda_\alpha \frac{\partial f_\alpha}{\partial q_j} \tag{2.16}$$

The lagrangian is

$$\mathcal{L} = \frac{1}{2}m\Big(\dot{r}^2 + r^2\dot{\theta}^2\Big) + mgr\cos(\theta) \tag{2.17}$$

For r

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} \tag{2.18}$$

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{r}}\Big) = m\ddot{r} \tag{2.19}$$

$$\frac{\partial \mathcal{L}}{\partial r} = mr\dot{\theta}^2 + mg\cos(\theta) \tag{2.20}$$

$$Q_r = \lambda\frac{\partial f}{\partial r} = \lambda \tag{2.21}$$

$$\boxed{m\ddot{r} - mr\dot{\theta}^2 - mg\cos(\theta) = \lambda} \tag{2.22}$$

For θ

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} \tag{2.23}$$

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\Big) = 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} \tag{2.24}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -mgr\sin(\theta) \tag{2.25}$$

$$Q_\theta = \lambda\frac{\partial f}{\partial \theta} = 0 \tag{2.26}$$

$$\boxed{2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} + mgr\sin(\theta) = 0} \tag{2.27}$$

Apply the contraint (2.15) to (2.27)

$$ml^2\ddot{\theta} + mgl\sin(\theta) = 0 \tag{2.28}$$

Now multiply both sides by $\dot{\theta}$

$$\ddot{\theta}\dot{\theta} + \frac{g}{l}\sin(\theta)\dot{\theta} = 0$$

$$\frac{1}{2}\frac{d}{dt}\Big(\dot{\theta}^2\Big) - \frac{g}{l}\frac{d}{dt}\Big(\cos(\theta)\Big) = 0$$

Upon integration

$$\frac{1}{2}\dot{\theta}^2 - \frac{g}{l}\cos(\theta) = C \tag{2.29}$$

Where C is a constant. At $t = 0$, $\theta = 0$ and $\dot{\theta}_0 = \omega$; thus,

$$\frac{1}{2}\omega^2 - \frac{g}{l} = C \tag{2.30}$$

$$\dot{\theta}^2 = \omega^2 + \frac{2g}{l}\Big[\cos(\theta) - 1\Big] \tag{2.31}$$

Using (2.15) and (2.31) in (2.22)

$$-ml\Big[\omega^2 + \frac{2g}{l}\Big(\cos(\theta) - 1\Big)\Big] - mg\cos(\theta) = \lambda \tag{2.32}$$

$$-ml\omega^2 - mg\Big[3\cos(\theta) - 2\Big] = \lambda \tag{2.33}$$

Now, in order for the string to go slack, the tension in the string needs to be 0; thus, we require $\lambda = 0$ (The tension pushes rather than pulls). This will occur when

$$3\cos(\theta) - 2 = -\frac{l\omega^2}{g}$$

$$\cos(\theta) = \frac{2}{3} - \frac{l\omega^2}{3g}$$

$$\theta_s(\omega) = \cos^{-1}\left(\frac{2}{3} - \frac{l\omega^2}{3g}\right) \tag{2.34}$$

Above θ_s, the string will be slack and the constraint will no longer apply. Suppose $\lambda > 0$, then we would need.

$$\theta > \cos^{-1}\left(\frac{2}{3} - \frac{l\omega^2}{3g}\right) \tag{2.35}$$

However, we must also have $\theta < \pi$ due to the restriction on $\cos^{-1}(x)$; thus,

$$\cos^{-1}\left(\frac{2}{3} - \frac{l\omega^2}{3g}\right) < \theta < \pi \tag{2.36}$$

Based on (2.36) and the symmetry in the plane of rotation, if

$$\omega > \sqrt{\frac{5g}{l}} \tag{2.37}$$

then the particle will just rotate around the point of attachement. Thus, for the string to go slack we must have

$$\boxed{0 < \omega < \sqrt{\frac{5g}{l}}} \tag{2.38}$$

if $\omega < 0$, then we will just be kicking the particle in the opposite direction; thus, (2.38) gives the allowed values for ω in order for a slack string to occur for some range of values for θ above $\theta_s(\omega)$. Finally, plugging in the bounds on ω into (2.34)

$$\boxed{\cos^{-1}\left(\frac{2}{3}\right) < \theta_s(\omega) < \pi} \tag{2.39}$$

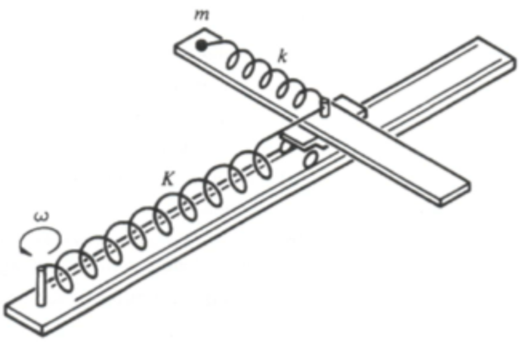
2.3 Problem 3

3. A carriage runs along rails on a rigid beam, as shown in the figure below. The carriage is attached to one end of a spring of equilibrium length r_0 and force constant k, whose other end is fixed on the beam. On the carriage, another set of rails is perpendicular to the first along which a particle of mass m moves, held by a spring fixed on the beam, of force constant k (the same as the first spring) and zero equilibrium length. Beam, rails, springs, and carriage area assumed to have zero mass. The whole system is forced to move in a plane about the point of attachment of the first spring, with a constant angular speed ω. The length of the first spring is at all times considered small compared to r_0.

a) What is the energy of the system? Is it conserved?

b) Using generalized coordinates in the laboratory frame, what is the Jacobi integral for the system? Is it conserved?

c) In terms of the generalized coordinates relative to a system rotating with the angular speed ω, what is the Lagrangian? What is the Jacobi integral? Is it conserved? Discuss the relationship between the two Jacobi integrals?



2.3.1 Part a)

Number of objects	N	1
Number of Dimensions	D	2
Number of translation D.O.F	$N_{\text{trans}} = ND$	2
Object dimensionality	C	0
Number of rotational D.O.F	$N_{\text{rot}} = \frac{1}{2}D(D-1) - \frac{1}{2}(D-C)(D-C-1)$	0
Number of constraints	M	0
Number of D.O.F	$f = N_{\text{trans}} + N_{\text{rot}} - M$	2

Choose G.Cs x', y' as G.Cs. Now $\theta = \omega t$ is the angle between the x-axis and the rail containing the cart; furthermore, let x' be the length of the spring of the attached from the origin to the cart and y' be the length of the sping attach from the cart to the particle. The prime coordinates are the G.Cs in the rotating frame.

The position vector of the particle in the rotating frame is

$$\mathfrak{r} = x'\hat{\mathbf{x}}' + y'\hat{\mathbf{y}}' \tag{2.40}$$

The primed frame is rotating clock-wise relative to the lab frame; thus,

$$\hat{\mathbf{x}}' = \cos(\omega t)\hat{\mathbf{x}} - \sin(\omega t)\hat{\mathbf{y}} \tag{2.41}$$

$$\hat{\mathbf{y}}' = \sin(\omega t)\hat{\mathbf{x}} + \cos(\omega t)\hat{\mathbf{y}} \tag{2.42}$$

So

$$\mathfrak{r} = (x'\cos(\theta) + y'\sin(\theta))\hat{\mathbf{x}} + (y'\cos(\theta) - x'\sin(\theta))\hat{\mathbf{y}} \tag{2.43}$$

Differentiating (2.43) with respect to time

$$\begin{aligned}\dot{\mathfrak{r}} = &\Big(\dot{x}'\cos(\theta) - x'\sin(\theta)\dot{\theta} + \dot{y}'\sin(\theta) + y'\cos(\theta)\dot{\theta}\Big)\hat{\mathbf{x}} \\ &+\Big(\dot{y}'\cos(\theta) - y'\sin(\theta)\dot{\theta} - \dot{x}'\sin(\theta) - x'\cos(\theta)\dot{\theta}\Big)\hat{\mathbf{y}}\end{aligned}$$

$$\begin{aligned}\dot{\mathfrak{r}}\cdot\dot{\mathfrak{r}} = \dot{x}'^2\cos^2(\theta) + x'^2\sin^2(\theta)\dot{\theta}^2 + \dot{y}'^2\sin^2(\theta) + y'^2\cos^2(\theta)\dot{\theta}^2\\ -2\dot{x}'x'\cos(\theta)\sin(\theta)\dot{\theta} + 2\dot{x}'\dot{y}'\cos(\theta)\sin(\theta) + 2\dot{x}'y'\cos^2(\theta)\dot{\theta}\\ -2x'\dot{y}'\sin^2(\theta)\dot{\theta} - 2x'y'\cos(\theta)\sin(\theta)\dot{\theta}^2\\ +2\dot{y}'y'\cos(\theta)\sin(\theta)\dot{\theta}\\ +\dot{y}'^2\cos^2(\theta) + y'^2\sin^2(\theta)\dot{\theta}^2 + \dot{x}'^2\sin^2(\theta) + x'^2\cos^2(\theta)\dot{\theta}^2\\ -2\dot{y}'y'\cos(\theta)\sin(\theta)\dot{\theta} - 2\dot{y}'\dot{x}'\cos(\theta)\sin(\theta) - 2\dot{y}'x'\cos^2(\theta)\dot{\theta}\\ +2y'\dot{x}'\sin^2(\theta)\dot{\theta} + 2y'x'\cos(\theta)\sin(\theta)\dot{\theta}^2\\ +2\dot{x}'x'\cos(\theta)\sin(\theta)\dot{\theta}\end{aligned}$$

$$\dot{\mathfrak{r}}\cdot\dot{\mathfrak{r}} = \dot{x}'^2+\dot{y}'^2+(x'^2+y'^2)\dot{\theta}^2+2\dot{x}'y'\dot{\theta}-2x'\dot{y}'\dot{\theta}+2(\dot{y}'y'+\dot{x}'x')\cos(\theta)\sin(\theta)\dot{\theta} \tag{2.44}$$

Thus, the kinetic energy is

$$\begin{aligned}T = \frac{1}{2}m\Big(\dot{x}'^2+\dot{y}'^2+(x'^2+y'^2)\omega^2\Big)\\ +\frac{1}{2}m\Big(\omega(\dot{y}'y'+\dot{x}'x')\sin(2\omega t)\Big) + m\omega\Big(\dot{x}'y'-x'\dot{y}'\Big)\end{aligned} \tag{2.45}$$

Now let $\mathfrak{r}_0 = x_0'\hat{\mathbf{x}}'$, then

$$\mathfrak{r}_0 = x_0'\Big(\cos(\theta)\hat{\mathbf{x}} - \sin(\theta)\hat{\mathbf{y}}\Big) \tag{2.46}$$

So

$$\mathfrak{r}-\mathfrak{r}_0 = ((x'-x_0')\cos(\theta)+y'\sin(\theta))\hat{\mathbf{x}} + (y'\cos(\theta)-(x'-x_0')\sin(\theta))\hat{\mathbf{y}} \tag{2.47}$$

$$(\mathfrak{r}-\mathfrak{r}_0)\cdot(\mathfrak{r}-\mathfrak{r}_0) = (x'-x_0')^2+y'^2 \tag{2.48}$$

The potential is

$$U = \frac{1}{2}m\Omega^2\Big((x'-x_0')^2+y'^2\Big) \tag{2.49}$$

Thus, the lagrangian in this frame is

$$\begin{aligned}\mathcal{L} = \frac{1}{2}m\Big(\dot{x}'^2+\dot{y}'^2+(x'^2+y'^2)\omega^2+\omega(\dot{y}'y'+\dot{x}'x')\sin(2\omega t)\Big)\\ +m\omega\Big(\dot{x}'y'-x'\dot{y}'\Big) - \frac{1}{2}m\Omega^2\Big((x'-x_0')^2+y'^2\Big)\end{aligned} \tag{2.50}$$

Furthermore, the total energy is

$$\begin{aligned}E = \frac{1}{2}m\Big(\dot{x}'^2+\dot{y}'^2+(x'^2+y'^2)\omega^2+\omega(\dot{y}'y'+\dot{x}'x')\sin(2\omega t)\Big)\\ +m\omega\Big(\dot{x}'y'-x'\dot{y}'\Big) + \frac{1}{2}m\Omega^2\Big((x'-x_0')^2+y'^2\Big)\end{aligned} \tag{2.51}$$

From (2.51), it is evident that the total energy is not conserved since $\frac{dE}{dt} \neq 0$

2.3.2 Part b)

Now for h

$$h = \Big[m\dot{x}'+\omega x'\sin(2\omega t)+m\omega y'\Big]\dot{x}'+\Big[m\dot{y}'+\omega y'\sin(2\omega t)-m\omega x'\Big]\dot{y}'-\frac{1}{2}m(x'^2+y'^2)\omega^2$$
$$-\frac{1}{2}m\Big(\dot{x}'^2+\dot{y}'^2+(x'^2+y'^2)\omega^2+\omega(\dot{y}'y'+\dot{x}'x')\sin(2\omega t)\Big)-m\omega\Big(\dot{x}'y'-x'\dot{y}'\Big)+\frac{1}{2}m\Omega^2\Big((x'-x_0')^2+y'^2\Big)$$

$$\boxed{\begin{aligned} h = {} & \frac{1}{2}m\Big(\dot{x}'^2+\dot{y}'^2\Big)+\frac{1}{2}m\Omega^2\Big((x'-x_0')^2+y'^2\Big) \\ & -\frac{1}{2}m(x'^2+y'^2)\omega^2+\frac{1}{2}m\omega(\dot{y}'y'+\dot{x}'x')\sin(2\omega t)\end{aligned}} \tag{2.52}$$

From (2.50) the lagrangian does explicitly depend on time; thus, h is not conserved.

We can also see upon comparing (2.52) to (2.51) that h is not the total energy.

2.3.3 Part c)

Now the position vector in the fixed frame is

$$\mathfrak{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} \tag{2.53}$$

Where

$$x = x'\cos(\omega t) + y'\cos\left(\omega t + \frac{\pi}{2}\right) \tag{2.54}$$

$$y = x'\sin(\omega t) + y'\sin\left(\omega t + \frac{\pi}{2}\right) \tag{2.55}$$

So

$$x = x'\cos(\omega t) - y'\sin(\omega t) \tag{2.56}$$

$$y = x'\sin(\omega t) + y'\cos(\omega t) \tag{2.57}$$

Taking the time derivatives of (2.56) and (2.57)

$$\dot{x} = \dot{x}'\cos(\omega t) - x'\sin(\omega t)\omega - \dot{y}'\sin(\omega t) - y'\cos(\omega t)\omega \tag{2.58}$$

$$\dot{y} = \dot{x}'\sin(\omega t) + x'\cos(\omega t)\omega + \dot{y}'\cos(\omega t) - y'\sin(\omega t)\omega \tag{2.59}$$

Squaring the velocities

$$\begin{gathered}\dot{x}^2 = \dot{x'}^2\cos^2(\omega t) + x'^2\sin^2(\omega t)\omega^2 + \dot{y'}^2\sin^2(\omega t) + y'^2\cos^2(\omega t)\omega^2\\ -2\dot{x'}x'\cos(\omega t)\sin(\omega t)\omega - 2\dot{x'}\dot{y'}\cos(\omega t)\sin(\theta) - 2\dot{R}r\cos^2(\theta)\dot{\theta}\\ +2x'\dot{y'}\sin^2(\omega t)\dot{\omega t} + 2x'y'\cos(\omega t)\sin(\omega t)\omega^2\\ +2\dot{y'}y'\cos(\omega t)\sin(\omega t)\omega\end{gathered}$$

$$\begin{gathered}\dot{y}^2 = \dot{x'}^2\sin^2(\omega t) + x'^2\cos^2(\omega t)\omega^2 + \dot{y'}^2\cos^2(\omega t) + y'^2\sin^2(\omega t)\omega^2\\ +2\dot{x'}x'\cos(\omega t)\sin(\omega t)\omega + 2\dot{x'}\dot{y'}\cos(\omega t)\sin(\omega) - 2\dot{x'}y'\sin^2(\omega t)\omega\\ +2x'\dot{y'}\cos^2(\omega t)\omega - 2x'y'\cos(\omega t)\sin(\omega t)\omega^2\\ -2\dot{y'}y'\cos(\omega t)\sin(\omega t)\omega\end{gathered}$$

Thus

$$T = \frac{1}{2}m\Big(\dot{x'}^2 + \dot{y'}^2 + (x'^2 + y'^2)\omega^2\Big) + m\Big(x'\dot{y'}\omega - \dot{x'}y'\omega\Big) \tag{2.60}$$

The potential energy is simply

$$U = \frac{1}{2}m\Omega^2\Big((x' - x'_0)^2 + y'^2\Big) \tag{2.61}$$

Where

$$m\Omega^2 = k \tag{2.62}$$

The Lagrangian is

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}m\Big(\dot{x'}^2 + \dot{y'}^2 + (x'^2 + y'^2)\omega^2\Big)\\ &+m\omega\Big(x'\dot{y'} - \dot{x'}y'\Big) - \frac{1}{2}m\Omega^2\Big((x' - x'_0)^2 + y'^2\Big)\end{aligned} \tag{2.63}$$

The total energy of the system is just

$$\begin{aligned}E' &= \frac{1}{2}m\Big(\dot{x'}^2 + \dot{y'}^2 + (x'^2 + y'^2)\omega^2\Big)\\ &+m\omega\Big(x'\dot{y'} - \dot{x'}y'\Big) + \frac{1}{2}m\Omega^2\Big((x' - x'_0)^2 + y'^2\Big)\end{aligned} \tag{2.64}$$

Now, the energy function is

$$h' = \sum_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\dot{q}_j - \mathcal{L} \tag{2.65}$$

So

$$\begin{gathered}h' = \Big[m\dot{x'} - m\omega y'\Big]\dot{x'} + \Big[m\dot{y'} + m\omega x'\Big]\dot{y'}\\ -\frac{1}{2}m\Big(\dot{x'}^2 + \dot{y'}^2 + (x'^2 + y'^2)\omega^2\Big) - m\omega\Big(x'\dot{y'} - \dot{x'}y'\Big) + \frac{1}{2}m\Omega^2\Big((x' - x'_0)^2 + y'^2\Big)\end{gathered}$$

$$(2.66) \quad h' = \frac{1}{2}m\left(\dot{x'}^2 + \dot{y'}^2\right) + \frac{1}{2}m\Omega^2\left((x' - x'_0)^2 + y'^2\right) - \frac{1}{2}m(x'^2 + y'^2)\omega^2$$

From (2.63) notice that $\frac{\partial \mathcal{L}}{\partial t} = 0$; thus h' is conservered. However we see that (2.64) and (2.66) are different; thus, h' is not the total energy of the system.

Furthermore, notice that h and h' are related via:

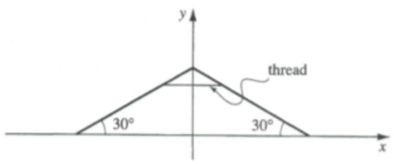
$$(2.67) \qquad h = h' + \frac{1}{2}m\omega\left(\dot{y}'y' + \dot{x}'x'\right)\sin(2\omega t)$$

Chapter 3

Homework 3

3.1 Problem 1

1. Two thin rods, each of mass *m* and length *l*, are connected to an ideal (frictionless) hinge and a horizontal thread. The system rests on a smooth surface as shown in the figure. At time *t = 0*, the thread is cut. Neglect the mass of the hinge and the thread and consider only the motion in the *xy* plane.



(a) Find the speed with which the hinge hits the floor.

(b) Find the time it takes for the hinge to hit the floor.

3.1.1 Part a)

Number of objects	N	2
Number of Dimensions	D	2
Number of translation D.O.F	$N_{\text{trans}} = ND$	4
Object dimensionality	C	1
Number of rotational D.O.F	$N_{\text{rot}} = \frac{N}{2}D(D-1) - \frac{N}{2}(D-C)(D-C-1)$	2
Number of constraints	M	5
Number of D.O.F	$f = N_{\text{trans}} + N_{\text{rot}} - M$	1

Let θ be the angle between the y-axis and one of the rods

$$T = \frac{1}{2}\left(ml^2\right)\dot{\theta}^2 \tag{3.1}$$

$$U = mgl\cos(\theta) \tag{3.2}$$

$$ml^2\ddot{\theta} + mgl\sin(\theta) = 0 \tag{3.3}$$

$$ml^2\ddot{\theta}\dot{\theta} - mgl\sin(\theta)\dot{\theta} = 0 \tag{3.4}$$

$$\frac{ml^2}{2}\frac{d}{dt}\Big(\dot{\theta}^2\Big) + mgl\frac{d}{dt}\Big(\cos(\theta)\Big) = 0 \tag{3.5}$$

$$\frac{ml^2}{2}\dot{\theta}^2 + mgl\cos(\theta) = C \tag{3.6}$$

Initially, $\theta(0) = \frac{\pi}{3}$ and $\dot{\theta}\Big|_{t=0} = 0$; thus,

$$\frac{mgl}{2} = C \tag{3.7}$$

Now,

$$\frac{ml^2}{2}\dot{\theta}^2 + mgl\cos(\theta) = \frac{mgl}{2} \tag{3.8}$$

When the rods hit the ground at t_f, then $\theta(t_f) = \frac{\pi}{2}$; thus,

$$\frac{ml^2}{2}\left[\dot{\theta}\Big|_{t=t_f}\right]^2 = \frac{mgl}{2} \tag{3.9}$$

$$\left[\dot{\theta}\Big|_{t=t_f}\right]^2 = \frac{g}{l} \tag{3.10}$$

So the final velocity of the hinge will be

$$\boxed{v_f = l\sqrt{\frac{g}{l}} = \sqrt{gl}} \tag{3.11}$$

3.1.2 Part b)

$$\begin{aligned} \triangle\theta &= \omega_0 t + \frac{1}{2}\alpha t^2 \\ \omega^2 &= \omega_0^2 + 2\alpha\triangle\theta \end{aligned} \tag{3.12}$$

Since $\omega_0 = 0$

$$\begin{aligned} \triangle\theta &= \frac{1}{2}\alpha t^2 \\ \omega^2 &= 2\alpha\triangle\theta \end{aligned} \tag{3.13}$$

$$\frac{\omega^2}{2\alpha} = \frac{\alpha t^2}{2} \tag{3.14}$$

$$\omega^2 = \alpha^2 t^2 \tag{3.15}$$

From (3.3)

$$\ddot{\theta}\Big|_{t=t_f} + \frac{g}{l}\sin\left(\frac{\pi}{2}\right) = 0 \tag{3.16}$$

$$\ddot{\theta}\Big|_{t=t_f} = -\frac{g}{l} \tag{3.17}$$

Now

$$t^2 = \left(\frac{\omega}{\alpha}\right)^2 \tag{3.18}$$

$$t^2 = \frac{g}{l}\left(-\frac{l}{g}\right)^2 \tag{3.19}$$

$$\boxed{t = \sqrt{\frac{l}{g}}} \tag{3.20}$$

3.2 Problem 2

2. (a) Express in terms of Euler's angles and the Cartesian coordinates of the center of mass the constraint conditions for a uniform sphere rolling without slipping on a flat horizontal surface. Show that they are nonholonomic.

(b) Set up the Lagrange equations for this problem *(Hint: take a look at p. 47 of the 3rd edition of Goldstein on semi-holonomic constraints)* by the method of Lagrange multipliers. You do NOT need to solve these equations. Are there any other constants of motion besides the rotational and translational components of the kinetic energy?

3.2.1 Part a)

Number of objects	N	1
Number of Dimensions	D	3
Number of translation D.O.F	$N_{\text{trans}} = ND$	3
Object dimensionality	C	3
Number of rotational D.O.F	$N_{\text{rot}} = \frac{N}{2}D(D-1) - \frac{N}{2}(D-C)(D-C-1)$	3
Number of constraints	M	3
Number of D.O.F	$f = N_{\text{trans}} + N_{\text{rot}} - M$	3

There are 3 constraints: we require that $\mathbf{v}_{\text{fixed}} = \mathbf{0}$ which means that $\mathbf{0} = \mathbf{v}_{\text{body}} + \boldsymbol{\omega} \times \mathbf{r}$ at the point of the sphere which has contact with the plane.

Let X, Y, and Z be the center of mass coordinates of the sphere with radius R; furthermore, let θ, ϕ, and ψ be the respective rotations about the $\hat{\mathbf{e}}_3$, $\hat{\boldsymbol{\rho}}_1$, and $\hat{\mathbf{e}}_3'$ axes. Now, from (A.28) for $\boldsymbol{\omega}$

$$\begin{aligned} \mathbf{v}_{\text{body}} &= \dot{X}\hat{\mathbf{x}} + \dot{Y}\hat{\mathbf{y}} + \dot{Z}\hat{\mathbf{z}} \\ \boldsymbol{\omega} &= \Big(\dot{\phi}\cos(\theta) - \dot{\psi}\sin(\phi)\sin(\theta)\Big)\hat{\mathbf{x}} + \Big(\dot{\phi}\sin(\theta) + \dot{\psi}\sin(\phi)\cos(\theta)\Big)\hat{\mathbf{y}} + \Big(\dot{\theta} + \dot{\psi}\cos(\phi)\Big)\hat{\mathbf{z}} \\ \mathbf{r} &= R\hat{\mathbf{z}} \end{aligned} \tag{3.21}$$

$$\begin{aligned} 0 &= \dot{X} + R\Big(\dot{\phi}\sin(\theta) + \dot{\psi}\sin(\phi)\cos(\theta)\Big) \\ 0 &= \dot{Y} - R\Big(\dot{\phi}\cos(\theta) - \dot{\psi}\sin(\phi)\sin(\theta)\Big) \\ 0 &= \dot{Z} + \Big(0\Big) \end{aligned} \tag{3.22}$$

ψ_0 is some constant.

Clearly, (3.22) are non-holonomic since we can't write them them as $f_\alpha(q_j) = 0$

3.2.2 Part b)

$$T_{\text{trans}} = \frac{1}{2}M\Big(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2\Big) \tag{3.23}$$

$$T_{\text{rot}} = \frac{1}{2}\Big(\boldsymbol{\omega}^T\mathbf{I}\boldsymbol{\omega}\Big) \tag{3.24}$$

From (A.43) we have that $\mathbf{I} = \frac{2}{5}MR^2\mathbf{1}$; thus, the rotational kinetice energy is just

$$T_{\text{rot}} = \frac{MR^2}{5}\omega^2 \tag{3.25}$$

From (A.22)

$$T_{\text{rot}} = \frac{MR^2}{5}\Big(\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 + 2\dot{\psi}\dot{\phi}\cos(\phi)\Big) \tag{3.26}$$

So the kinetic energy is just

$$T = \frac{1}{2}M\Big(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2\Big) + \frac{MR^2}{5}\Big(\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 + 2\dot{\psi}\dot{\theta}\cos(\phi)\Big) \tag{3.27}$$

The potential energy of the sphere is just

$$U = MgZ \tag{3.28}$$

Thus, the lagrangian is simply

$$\mathcal{L} = \frac{1}{2}M\Big(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2\Big) + \frac{MR^2}{5}\Big(\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 + 2\dot{\psi}\dot{\theta}\cos(\phi)\Big) - MgZ \tag{3.29}$$

From Goldstein (2.25) and (2.27) respectively

$$\begin{aligned} f_\alpha &= \sum_{k=1}^{n} a_{\alpha k}\dot{q}_k + a_0 \\ \frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\Big) - \frac{\partial \mathcal{L}}{\partial q_j} &= -\sum_{\alpha=1}^{m} \mu_\alpha(t)\frac{\partial f_\alpha}{\partial \dot{q}_j} \end{aligned} \tag{3.30}$$

For X:

$$M\ddot{X} = -\mu_1(t) \tag{3.31}$$

For Y:

$$M\ddot{Y} = -\mu_2(t) \tag{3.32}$$

For Z:

$$M\ddot{Z} + Mg = -\mu_3(t) \tag{3.33}$$

For θ:

$$\frac{2MR^2}{5}\Big(\ddot{\theta} + \ddot{\psi}\cos(\phi) - \dot{\psi}\sin(\phi)\dot{\phi}\Big) = 0 \tag{3.34}$$

For ϕ:

$$\frac{2MR^2}{5}\Big(\ddot{\phi} + \dot{\psi}\dot{\theta}\sin(\phi)\Big) = -R\Big[\mu_1(t)\sin(\theta) - \mu_2(t)\cos(\theta)\Big] \tag{3.35}$$

For ψ:

$$\frac{2MR^2}{5}\Big(\ddot{\psi} + \ddot{\theta}\cos(\phi) - \dot{\theta}\sin(\phi)\dot{\phi}\Big) = -R\sin(\phi)\Big[\mu_1(t)\cos(\theta) - \mu_2(t)\sin(\theta)\Big] \tag{3.36}$$

From (3.33) we can see that

$$\frac{2MR^2}{5}\left(\dot{\theta} + \dot{\psi}\cos(\phi)\right) = \text{const} \tag{3.37}$$

3.3 Problem 3

3. Consider a symmetric top *($I_1 = I_2 \neq I_3$)* with constant charge-to-mass ratio *q/m* and hence constant gyromagnetic ratio *γ = q/2m.* As discussed in lecture, such an object will have a Lagrangian:

$L = T - V = T - \boldsymbol{\omega}_l \bullet \boldsymbol{L}$

where $\boldsymbol{\omega}_l = -\gamma \boldsymbol{B}$ is the Larmor frequency and ***L*** is the angular momentum.

(a) Show that the kinetic energy *T* is constant and find the other constants of motion.

(b) Under the assumption that ω_l is much less than the initial component of the angular velocity along the symmetry axis, obtain expressions for the frequency and amplitude of nutation and the average precession frequency.

3.3.1 Part a)

For a symmetric top

$$I_1 = I_2 \neq I_3 \tag{3.38}$$

Using the body frame, the kinetic energy is using ϕ, θ, and ψ as the 3 Euler angles.

$$T = \frac{1}{2}I_1\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2(\theta)\right) + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos(\theta))^2 \tag{3.39}$$

The potential energy is

$$V = \boldsymbol{\omega}_l \cdot \mathbf{L} \tag{3.40}$$

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega} \tag{3.41}$$

where $\mathbf{I}$ is the inertia tensor: in the body frame

$$\mathbf{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \tag{3.42}$$

So, using $\boldsymbol{\omega}$ in the body frame:

$$\begin{aligned} \mathbf{L} = I_1\hat{\mathbf{e}}_1'&\Big(\dot{\phi}\sin(\theta)\sin(\psi) + \dot{\theta}\cos(\psi)\Big) \\ +I_2\hat{\mathbf{e}}_2'&\Big(\dot{\phi}\sin(\theta)\cos(\psi) - \dot{\theta}\sin(\psi)\Big) \\ +I_3\hat{\mathbf{e}}_3'&\Big(\dot{\psi} + \dot{\phi}\cos(\theta)\Big) \end{aligned} \tag{3.43}$$

Since $I_1 = I_2$

$$\begin{aligned} \mathbf{L} = I_1\hat{\mathbf{e}}_1'&\Big(\dot{\phi}\sin(\theta)\sin(\psi) + \dot{\theta}\cos(\psi)\Big) \\ I_1\hat{\mathbf{e}}_2'&\Big(\dot{\phi}\sin(\theta)\cos(\psi) - \dot{\theta}\sin(\psi)\Big) \\ +I_3\hat{\mathbf{e}}_3'&\Big(\dot{\psi} + \dot{\phi}\cos(\theta)\Big) \end{aligned} \tag{3.44}$$

$$\boldsymbol{\omega}_l = -\gamma\mathbf{B} \tag{3.45}$$

Suppose that $\mathbf{B}$ is oriented along the fixed z axis

$$\mathbf{B} = B\hat{\mathbf{e}}_z \tag{3.46}$$

expressed in the body frame.

$$\mathbf{B} = B\Big(\sin(\theta)\sin(\psi)\Big)\hat{\mathbf{e}}_1' + B\Big(\sin(\theta)\cos(\psi)\Big)\hat{\mathbf{e}}_2' + B\cos(\theta)\hat{\mathbf{e}}_3' \tag{3.47}$$

Now the potential energy is simply

$$\begin{aligned} V &= -\gamma\mathbf{B}\cdot\mathbf{L} \\ V &= -\gamma B\Big(I_1\dot{\phi}\sin^2(\theta) + I_3\cos(\theta)\Big(\dot{\psi} + \dot{\phi}\cos(\theta)\Big)\Big) \end{aligned} \tag{3.48}$$

So the Lagrangian is just

$$\begin{aligned} \mathcal{L} = \frac{1}{2}I_1\Big(\dot{\theta}^2 + \dot{\phi}^2\sin^2(\theta)\Big) + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos(\theta))^2 \\ +\gamma B\Big(I_1\dot{\phi}\sin^2(\theta) + I_3\cos(\theta)\Big(\dot{\psi} + \dot{\phi}\cos(\theta)\Big)\Big) \end{aligned} \tag{3.49}$$

Since ϕ and ψ are not present in the lagrangian: they are cyclic coordinates; thus, there are two conserved quantitities

For ϕ

$$\begin{aligned} I_1\dot{\phi}\sin^2(\theta) + I_3(\dot{\psi} + \dot{\phi}\cos(\theta))\cos(\theta) \\ +\gamma B\Big(I_1\sin^2(\theta) + I_3\cos^2(\theta)\Big) = I_1 b \\ \text{where } b \text{ is a constant} \end{aligned} \tag{3.50}$$

For ψ

$$I_3(\dot{\psi} + \dot{\phi}\cos(\theta)) + \gamma B I_3 \cos(\theta) = I_1 a \quad \text{where } a \text{ is a constant} \tag{3.51}$$

Solving for $\dot{\phi}$

$$I_1 \dot{\phi} \sin^2(\theta) + \gamma B I_1 \sin^2(\theta) = I_1(b - a\cos(\theta)) \tag{3.52}$$

$$\dot{\phi} = \frac{b - a\cos(\theta) - \gamma B \sin^2(\theta)}{\sin^2(\theta)} \tag{3.53}$$

And $\dot{\psi}$

$$\dot{\psi} = \frac{I_1 a}{I_3} - \Big[\frac{b - a\cos(\theta) - \gamma B \sin^2(\theta)}{\sin^2(\theta)} + \gamma B\Big]\cos(\theta) \tag{3.54}$$

From (3.48)

$$V = -\gamma B\Big(I_1\Big[b - a\cos(\theta) - \gamma B\sin^2(\theta)\Big] + I_1 a\cos(\theta) - \gamma I_3 B\cos^2(\theta)\Big) \tag{3.55}$$

$$V = -\gamma B\Big(I_1\Big[b - \gamma B\sin^2(\theta)\Big] - \gamma I_3 B\cos^2(\theta)\Big) \tag{3.56}$$

$$V = -\gamma B I_1 b + \gamma^2 I_1 B^2 \sin^2(\theta) + \gamma^2 I_3 B^2 \cos^2(\theta) \tag{3.57}$$

$$V = -\gamma B I_1 b + \gamma^2 I_1 B^2 + \gamma^2 (I_3 - I_1) B^2 \cos^2(\theta) \tag{3.58}$$

Now making the same substitutions into the kinetic energy:

$$T = \frac{1}{2}I_1\Big(\dot{\theta}^2 + \Big[\frac{b - a\cos(\theta)}{\sin^2(\theta)} - \gamma B\Big]^2 \sin^2(\theta)\Big) + \frac{1}{2}I_3\Big(\frac{I_1 a}{I_3} - \gamma B\cos(\theta)\Big)^2 \tag{3.59}$$

(3.60)

$$T = \frac{1}{2}I_1\Big(\dot{\theta}^2 + \frac{(b - a\cos(\theta))^2}{\sin^2(\theta)} - b\gamma B + \gamma^2 B^2 \sin^2(\theta)\Big) + \frac{1}{2}I_3\Big(\frac{I_1^2 a^2}{I_3^2} + \gamma^2 B^2\cos^2(\theta)\Big)$$

(3.61)

$$T = \frac{1}{2}\frac{I_1^2 a^2}{I_3} + \frac{1}{2}I_1\gamma^2 B^2 - \frac{1}{2}I_1 b\gamma B + \frac{1}{2}I_1\dot{\theta}^2 + \frac{I_1\gamma B(b - a\cos(\theta))^2}{2\sin^2(\theta)} + \frac{1}{2}\gamma^2(1_3 - I_1)B^2\cos^2(\theta)$$

Since we have constant terms in both T and V we can redefine the kinetic and potential energies as

$$V' = \gamma^2 B^2 (I_3 - I_1)\cos^2(\theta) \tag{3.62}$$

and

$$T' = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{I_1 \gamma B (b - a\cos(\theta))^2}{\sin^2(\theta)} + \frac{1}{2}\gamma^2 B^2 (I_3 - I_1)\cos^2(\theta) \tag{3.63}$$

$$\mathcal{L}' = T' - V'$$
$$\mathcal{L}' = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{I_1 \gamma B (b - a\cos(\theta))^2}{\sin^2(\theta)} + \frac{3}{2}\gamma^2 B^2 (I_3 - I_1)\cos^2(\theta) \tag{3.64}$$

<u>For θ</u>

(3.65)
$$I_1\ddot{\theta} - \frac{2I_1\gamma Ba(b - a\cos(\theta))\sin^2(\theta) - 2I_1\gamma B(b - a\cos(\theta))^2}{\sin^3(\theta)} + 3\gamma^2 B^2 (I_3 - I_1)\cos(\theta)\sin(\theta) = 0$$

Taking the full time derivative of the kinetic energy because we wish to show that T' is a constant.

(3.66)
$$\frac{dT'}{dt} = \frac{\partial T'}{\partial \dot{\theta}}\ddot{\theta} + \frac{\partial T'}{\partial \theta}\dot{\theta} + \frac{\partial T'}{\partial t}$$
$$\Big[I_1\ddot{\theta} - \frac{2I_1\gamma Ba(b - a\cos(\theta))\sin^2(\theta) - 2I_1\gamma B(b - a\cos(\theta))^2}{\sin^3(\theta)} + \gamma^2 B^2 (I_3 - I_1)\cos(\theta)\sin(\theta)\Big]\dot{\theta} = 0$$

From (3.65) we can add subtract $2\gamma^2 B^2 (I_3 - I_1)\cos(\theta)\sin(\theta)$ to (3.66), then

$$\Big[-2\gamma^2 B^2 (I_3 - I_1)\cos(\theta)\sin(\theta)\Big]\dot{\theta} = 0 \tag{3.67}$$

3.3.2 Part b)

Appendix A

Appendix

A.1 Spherical Coordinates

A.1.1 Velocity in Spherical Coordinates

$$\begin{cases} x = r\cos(\phi)\sin(\theta) \\ y = r\sin(\phi)\sin(\theta) \\ z = r\cos(\theta) \end{cases}$$

So

$$\dot{x} = \dot{r}\cos(\phi)\sin(\theta) - r\sin(\phi)\dot{\phi}\sin(\theta) + r\cos(\phi)\cos(\theta)\dot{\theta} \tag{A.1}$$

$$\dot{y} = \dot{r}\sin(\phi)\sin(\theta) + r\cos(\phi)\dot{\phi}\sin(\theta) + r\sin(\phi)\cos(\theta)\dot{\theta} \tag{A.2}$$

$$\dot{z} = \dot{r}\cos(\theta) - r\sin(\theta)\dot{\theta} \tag{A.3}$$

A.1.2 Square of Velocity in Spherical Coordinates

$$\begin{gathered}
\dot{x}^2 = \dot{r}^2\cos^2(\phi)\sin^2(\theta) + r^2\sin^2(\phi)\dot{\phi}^2\sin^2(\theta) + r^2\cos^2(\phi)\cos^2(\theta)\dot{\theta}^2 \\
-2\dot{r}\cos(\phi)\sin(\theta)r\sin(\phi)\dot{\phi}\sin(\theta) + 2\dot{r}\cos(\phi)\sin(\theta)r\cos(\phi)\cos(\theta)\dot{\theta} \\
-2r\sin(\phi)\dot{\phi}\sin(\theta)r\cos(\phi)\cos(\theta)\dot{\theta} \\
\dot{y}^2 = \dot{r}^2\sin^2(\phi)\sin^2(\theta) + r^2\cos^2(\phi)\dot{\phi}^2\sin^2(\theta) + r^2\sin^2(\phi)\cos^2(\theta)\dot{\theta}^2 \\
+2\dot{r}\sin(\phi)\sin(\theta)r\cos(\phi)\dot{\phi}\sin(\theta) + 2\dot{r}\sin(\phi)\sin(\theta)r\sin(\phi)\cos(\theta)\dot{\theta} \\
+2r\cos(\phi)\dot{\phi}\sin(\theta)r\sin(\phi)\cos(\theta)\dot{\theta} \\
\dot{z}^2 = \dot{r}^2\cos^2(\theta) + r^2\sin^2(\theta)\dot{\theta}^2 - 2\dot{r}\cos(\theta)r\sin(\theta)\dot{\theta}
\end{gathered}$$

$$\begin{gathered}
\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r}^2\sin^2(\theta) + r^2\sin^2(\theta)\dot{\phi}^2 + r^2\cos^2(\theta)\dot{\theta}^2 \\
+\dot{r}^2\cos^2(\theta) + r^2\sin^2(\theta)\dot{\theta}^2 \\
2\dot{r}\cos^2(\phi)\sin(\theta)r\cos(\theta)\dot{\theta} + 2\dot{r}\sin^2(\phi)\sin(\theta)r\cos(\theta)\dot{\theta} - 2\dot{r}\cos(\theta)r\sin(\theta)\dot{\theta}
\end{gathered}$$

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r}^2 + r^2\sin^2(\theta)\dot{\phi}^2 + r^2\dot{\theta}^2 \tag{A.4}$$

A.1.3 Time Derivative of the Position Vector in Spherical Coordinates

$$\mathbf{R} = R\hat{\mathbf{r}}(\Phi, \Theta)$$

$$\dot{\mathbf{R}} = \dot{R}\hat{\mathbf{r}} + R\Big[\frac{\partial \hat{\mathbf{r}}}{\partial \Phi}\dot{\Phi} + \frac{\partial \hat{\mathbf{r}}}{\partial \Theta}\dot{\Theta}\Big]$$

Now

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos(\Phi)\sin(\Theta) + \hat{\mathbf{y}}\sin(\Phi)\sin(\Theta) + \hat{\mathbf{z}}\cos(\Theta)$$

$$\frac{\partial \hat{\mathbf{r}}}{\partial \Phi} = -\hat{\mathbf{x}}\sin(\Phi)\sin(\Theta) + \hat{\mathbf{y}}\cos(\Phi)\sin(\Theta)$$

$$\Big|\frac{\partial \hat{\mathbf{r}}}{\partial \Phi}\Big| = \sin(\Theta)$$

So,

$$\frac{1}{\sin(\Theta)}\frac{\partial \hat{\mathbf{r}}}{\partial \Phi} = \hat{\boldsymbol{\phi}}$$

$$\frac{\partial \hat{\mathbf{r}}}{\partial \Theta} = \hat{\mathbf{x}}\cos(\Phi)\cos(\Theta) + \hat{\mathbf{y}}\sin(\Phi)\cos(\Theta) - \hat{\mathbf{z}}\sin(\Theta)$$

$$\Big|\frac{\partial \hat{\mathbf{r}}}{\partial \Theta}\Big| = 1$$

$$\frac{\partial \hat{\mathbf{r}}}{\partial \Theta} = \hat{\boldsymbol{\theta}}$$

Thus,

$$\dot{\mathbf{R}} = \hat{\mathbf{r}}\dot{R} + \hat{\boldsymbol{\phi}}R\sin(\Theta)\dot{\Phi} + \hat{\boldsymbol{\theta}}R\dot{\Theta} \tag{A.5}$$

A.1.4 Gradient in Spherical Coordinates

$$\nabla f = \sum_{i=1}^{N} \underbrace{\frac{1}{h_i}\frac{\partial \mathbf{r}}{\partial q_i}}_{\hat{\mathbf{e}}_i}\frac{\partial f}{\partial q_i}$$

$$\frac{\partial \mathbf{r}}{\partial r} = \hat{\mathbf{x}}\cos(\phi)\sin(\theta) + \hat{\mathbf{y}}\sin(\phi)\sin(\theta) + \hat{\mathbf{z}}\cos(\theta)$$

$$h_r = \Big|\frac{\partial \mathbf{r}}{\partial r}\Big| = 1$$

So,

$$\hat{\mathbf{e}}_r = \hat{\mathbf{x}}\cos(\phi)\sin(\theta) + \hat{\mathbf{y}}\sin(\phi)\sin(\theta) + \hat{\mathbf{z}}\cos(\theta) \tag{A.6}$$

$$\frac{\partial \mathbf{r}}{\partial \phi} = -\hat{\mathbf{x}}r\sin(\phi)\sin(\theta) + \hat{\mathbf{y}}r\cos(\phi)\sin(\theta)$$

$$h_\phi = \left|\frac{\partial \mathbf{r}}{\partial \phi}\right| = r\sin(\theta)$$

(A.7) $$\hat{\mathbf{e}}_\phi = -\hat{\mathbf{x}}\sin(\phi) + \hat{\mathbf{y}}\cos(\phi)$$

$$\frac{\partial \mathbf{r}}{\partial \theta} = \hat{\mathbf{x}}r\cos(\phi)\cos(\theta) + \hat{\mathbf{y}}r\sin(\phi)\cos(\theta) - \hat{\mathbf{z}}r\sin(\theta)$$

$$h_\theta = \left|\frac{\partial \mathbf{r}}{\partial \theta}\right| = r$$

(A.8) $$\hat{\mathbf{e}}_\theta = \hat{\mathbf{x}}\cos(\phi)\cos(\theta) + \hat{\mathbf{y}}\sin(\phi)\cos(\theta) - \hat{\mathbf{z}}\sin(\theta)$$

Thus, in spherical coordinates

(A.9) $$\nabla f = \hat{\mathbf{e}}_r\frac{\partial f}{\partial r} + \frac{\hat{\mathbf{e}}_\phi}{r\sin(\theta)}\frac{\partial f}{\partial \phi} + \frac{\hat{\mathbf{e}}_\theta}{r}\frac{\partial f}{\partial \theta}$$

A.1.5 Relation between Spherical and Cartesian

$$\begin{bmatrix}\hat{\mathbf{e}}_r\\ \hat{\mathbf{e}}_\phi\\ \hat{\mathbf{e}}_\theta\end{bmatrix} = \begin{bmatrix}\cos(\phi)\sin(\theta) & \sin(\phi)\sin(\theta) & \cos(\theta)\\ -\sin(\phi) & \cos(\phi) & 0\\ \cos(\phi)\cos(\theta) & \sin(\phi)\cos(\theta) & -\sin(\theta)\end{bmatrix}\begin{bmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{y}}\\ \hat{\mathbf{z}}\end{bmatrix}$$

$$\begin{bmatrix}\cos(\phi)\sin(\theta) & -\sin(\phi) & \cos(\phi)\cos(\theta)\\ \sin(\phi)\sin(\theta) & \cos(\phi) & \sin(\phi)\cos(\theta)\\ \cos(\theta) & 0 & -\sin(\theta)\end{bmatrix}\begin{bmatrix}\hat{\mathbf{e}}_r\\ \hat{\mathbf{e}}_\phi\\ \hat{\mathbf{e}}_\theta\end{bmatrix} = \begin{bmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{y}}\\ \hat{\mathbf{z}}\end{bmatrix}$$

(A.10) $$\hat{\mathbf{x}} = \hat{\mathbf{e}}_r\cos(\phi)\sin(\theta) - \hat{\mathbf{e}}_\phi\sin(\phi) + \hat{\mathbf{e}}_\theta\cos(\phi)\cos(\theta)$$

(A.11) $$\hat{\mathbf{y}} = \hat{\mathbf{e}}_r\sin(\phi)\sin(\theta) + \hat{\mathbf{e}}_\phi\cos(\phi) + \hat{\mathbf{e}}_\theta\sin(\phi)\cos(\theta)$$

(A.12) $$\hat{\mathbf{z}} = \hat{\mathbf{e}}_r\cos(\theta) - \hat{\mathbf{e}}_\theta\sin(\theta)$$

A.2 Euler Angles

Suppose θ is the rotation about the $\hat{\mathbf{e}}_3$, ϕ is the rotation about the $\hat{\boldsymbol{\rho}}_1$ axis, and ψ is the rotatation about the $\hat{\boldsymbol{\rho}}_3'$

A.2.1 Angular Velocity in the body frame

Now

$$\boldsymbol{\omega} = \dot{\theta}\hat{\mathbf{e}}_3 + \dot{\phi}\hat{\boldsymbol{\rho}}_1 + \dot{\psi}\hat{\boldsymbol{\rho}}_3' \tag{A.13}$$

Body Frame
Between the fixed frame and the $\boldsymbol{\rho}$ frame.

$$\begin{aligned} \hat{\mathbf{e}}_1 &= \cos(\theta)\hat{\boldsymbol{\rho}}_1 - \sin(\theta)\hat{\boldsymbol{\rho}}_2 \\ \hat{\mathbf{e}}_2 &= \sin(\theta)\hat{\boldsymbol{\rho}}_1 + \cos(\theta)\hat{\boldsymbol{\rho}}_2 \\ \hat{\mathbf{e}}_3 &= \hat{\boldsymbol{\rho}}_3 \end{aligned} \tag{A.14}$$

Thus,

$$\boldsymbol{\omega} = \dot{\theta}\hat{\rho}_3 + \dot{\phi}\hat{\boldsymbol{\rho}}_1 + \dot{\psi}\hat{\boldsymbol{\rho}}_3' \tag{A.15}$$

Between the $\boldsymbol{\rho}$ frame and the $\boldsymbol{\rho}'$ frame

$$\begin{aligned} \hat{\boldsymbol{\rho}}_1 &= \hat{\boldsymbol{\rho}}_1' \\ \hat{\boldsymbol{\rho}}_2 &= \cos(\phi)\hat{\boldsymbol{\rho}}_2' - \sin(\phi)\hat{\boldsymbol{\rho}}_3' \\ \hat{\boldsymbol{\rho}}_3 &= \sin(\phi)\hat{\boldsymbol{\rho}}_2' + \cos(\phi)\hat{\boldsymbol{\rho}}_3' \end{aligned} \tag{A.16}$$

So,

$$\boldsymbol{\omega} = \dot{\theta}\Big(\sin(\phi)\hat{\boldsymbol{\rho}}_2' + \cos(\phi)\hat{\boldsymbol{\rho}}_3'\Big) + \dot{\phi}\hat{\boldsymbol{\rho}}_1' + \dot{\psi}\hat{\boldsymbol{\rho}}_3' \tag{A.17}$$

Or Equivalently

$$\boldsymbol{\omega} = \dot{\theta}\sin(\phi)\hat{\boldsymbol{\rho}}_2' + \dot{\phi}\hat{\boldsymbol{\rho}}_1' + \Big(\dot{\psi} + \dot{\theta}\cos(\phi)\Big)\hat{\boldsymbol{\rho}}_3' \tag{A.18}$$

Between the $\boldsymbol{\rho}'$ frame and the body frame

$$\begin{aligned} \hat{\boldsymbol{\rho}}_1' &= \cos(\psi)\hat{\mathbf{e}}_1' - \sin(\psi)\hat{\mathbf{e}}_2' \\ \hat{\boldsymbol{\rho}}_2' &= \sin(\psi)\hat{\mathbf{e}}_1' + \cos(\psi)\hat{\mathbf{e}}_2' \\ \hat{\boldsymbol{\rho}}_3' &= \hat{\mathbf{e}}_3' \end{aligned} \tag{A.19}$$

$$\boldsymbol{\omega} = \dot{\theta}\sin(\phi)\Big(\sin(\psi)\hat{\mathbf{e}}_1' + \cos(\psi)\hat{\mathbf{e}}_2'\Big) + \dot{\phi}\Big(\cos(\psi)\hat{\mathbf{e}}_1' - \sin(\psi)\hat{\mathbf{e}}_2'\Big) + \Big(\dot{\psi} + \dot{\theta}\cos(\phi)\Big)\hat{\boldsymbol{\rho}}_3' \tag{A.20}$$

Or Equivalently

$$\boldsymbol{\omega} = \Big(\dot{\theta}\sin(\phi)\sin(\psi) + \dot{\phi}\cos(\psi)\Big)\hat{\mathbf{e}}_1' + \Big(\dot{\theta}\sin(\phi)\cos(\psi) - \dot{\phi}\sin(\psi)\Big)\hat{\mathbf{e}}_2' + \Big(\dot{\psi} + \dot{\theta}\cos(\phi)\Big)\hat{\mathbf{e}}_3' \tag{A.21}$$

Angular velocity squared in the body frame

$$\omega^2 = \dot\theta^2 + \dot\phi^2 + \dot\psi^2 + 2\dot\psi\dot\theta\cos(\phi) \tag{A.22}$$

A.3 Angular velocity in the fixed frame

$$\boldsymbol{\omega} = \dot\theta\hat{\mathbf{e}}_3 + \dot\phi\hat{\boldsymbol{\rho}}_1 + \dot\psi\hat{\boldsymbol{\rho}}'_3 \tag{A.23}$$

Between the $\boldsymbol{\rho}'$ and $\boldsymbol{\rho}$ frames

$$\begin{aligned} \hat{\boldsymbol{\rho}}'_1 &= \hat{\boldsymbol{\rho}}_1 \\ \hat{\boldsymbol{\rho}}'_2 &= \cos(\phi)\hat{\boldsymbol{\rho}}_2 - \sin(\phi)\hat{\boldsymbol{\rho}}_3 \\ \hat{\boldsymbol{\rho}}'_3 &= \sin(\phi)\hat{\boldsymbol{\rho}}_2 + \cos(\phi)\hat{\boldsymbol{\rho}}_3 \end{aligned} \tag{A.24}$$

So,

$$\boldsymbol{\omega} = \dot\theta\hat{\mathbf{e}}_3 + \dot\phi\hat{\boldsymbol{\rho}}_1 + \dot\psi\Big(\sin(\phi)\hat{\boldsymbol{\rho}}_2 + \cos(\phi)\hat{\boldsymbol{\rho}}_3\Big) \tag{A.25}$$

Between the $\boldsymbol{\rho}$ and the fixed frame

$$\begin{aligned} \hat{\boldsymbol{\rho}}_1 &= \cos(\theta)\hat{\mathbf{e}}_1 + \sin(\theta)\hat{\mathbf{e}}_2 \\ \hat{\boldsymbol{\rho}}_2 &= -\sin(\theta)\hat{\mathbf{e}}_1 + \cos(\theta)\hat{\mathbf{e}}_2 \\ \hat{\boldsymbol{\rho}}_3 &= \hat{\mathbf{e}}_3 \end{aligned} \tag{A.26}$$

Now,

$$\boldsymbol{\omega} = \dot\theta\hat{\mathbf{e}}_3 + \dot\phi\Big(\cos(\theta)\hat{\mathbf{e}}_1 + \sin(\theta)\hat{\mathbf{e}}_2\Big) + \dot\psi\sin(\phi)\Big(-\sin(\theta)\hat{\mathbf{e}}_1 + \cos(\theta)\hat{\mathbf{e}}_2\Big) + \dot\psi\cos(\phi)\hat{\mathbf{e}}_3 \tag{A.27}$$

Or Equivalently

$$\boldsymbol{\omega} = \Big(\dot\phi\cos(\theta) - \dot\psi\sin(\phi)\sin(\theta)\Big)\hat{\mathbf{e}}_1 + \Big(\dot\phi\sin(\theta) + \dot\psi\sin(\phi)\cos(\theta)\Big)\hat{\mathbf{e}}_2 + \Big(\dot\theta + \dot\psi\cos(\phi)\Big)\hat{\mathbf{e}}_3 \tag{A.28}$$

It can be checked that (A.22) holds true for $\boldsymbol{\omega}$ in the fixed frame as well.

A.4 Moments of Inertia

The inertia tensor at some point $\mathcal{P}$ is given by

$$\begin{aligned} J_{pq} &= \sum_k m_k\Big(r_k^2\delta_{pq} - x_{k,p}x_{k,q}\Big) \\ J_{pq} &= \int_V \rho(\mathbf{r})\Big(r^2\delta_{pq} - x_p x_q\Big)d\tau \end{aligned} \tag{A.29}$$

The first is for a discrete set of particles and the latter is the continuous case.

If we wish to get the inertia tensor at the center of mass, then suppose the separation vector between the center of mass and the original point $\mathcal{P}$ is given by $\mathbf{a}$; thus, by Steiner's parallel axis theorem.

$$I_{pq} = J_{pq} + M\Big(a^2\delta_{pq} - a_p a_q\Big) \tag{A.30}$$

For a solid sphere with constant mass density ρ about the center of mass:

$$I_{pq} = \rho \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{R} \Big(r^2\delta_{pq} - x_p x_q\Big) r^2 \sin(\theta) dr d\theta d\phi \tag{A.31}$$

When $p = q$

$$I_{xx} = \rho \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{R} \Big(r^2 - x^2\Big) r^2 \sin(\theta) dr d\theta d\phi \tag{A.32}$$

$$I_{xx} = \rho \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{R} \Big(r^2 - (r\cos(\phi)\sin(\theta))^2\Big) r^2 \sin(\theta) dr d\theta d\phi \tag{A.33}$$

$$I_{xx} = \rho \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{R} \Big(r^4 - r^4\cos^2(\phi)\sin^2(\theta)\Big) \sin(\theta) dr d\theta d\phi \tag{A.34}$$

$$I_{xx} = \frac{\rho R^5}{5} \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \Big(\sin(\theta) - \cos^2(\phi)\sin^2(\theta)\sin(\theta)\Big) d\theta d\phi \tag{A.35}$$

$$I_{xx} = \frac{\rho R^5}{5} \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \Big(\sin(\theta) - \cos^2(\phi)\Big(1 - \cos^2(\theta)\Big)\sin(\theta)\Big) d\theta d\phi \tag{A.36}$$

$$I_{xx} = \frac{\rho R^5}{5} \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \Big(\Big(1 - \cos^2(\phi)\Big)\sin(\theta) + \cos^2(\phi)\cos^2(\theta)\sin(\theta)\Big) d\theta d\phi \tag{A.37}$$

$$I_{xx} = \frac{\rho R^5}{5} \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \Big(\sin^2(\phi)\sin(\theta) + \cos^2(\phi)\cos^2(\theta)\sin(\theta)\Big) d\theta d\phi \tag{A.38}$$

$$I_{xx} = \frac{\rho R^5}{5} \int\limits_{\phi=0}^{2\pi} \Big(2\sin^2(\phi) + \frac{2}{3}\cos^2(\phi)\Big) d\phi \tag{A.39}$$

$$I_{xx} = \frac{\rho R^5}{5} \Big(2\pi + \frac{2}{3}\pi\Big) \tag{A.40}$$

$$I_{xx} = \frac{2MR^2}{5} \tag{A.41}$$

By symmetry

$$I_{xx} = I_{yy} = I_{zz} = \frac{2MR^2}{5} \tag{A.42}$$

Next I_{xy}

$$I_{xy} = \rho \int\limits_{\phi=0}^{2\pi} \int\limits_{\theta=0}^{\pi} \int\limits_{r=0}^{R} \Big(- xy\Big) r^2 \sin(\theta) dr d\theta d\phi \tag{A.43}$$

$$I_{xy} = \rho \int\limits_{\phi=0}^{2\pi} \int\limits_{\theta=0}^{\pi} \int\limits_{r=0}^{R} \Big(- r^2 \sin^2(\theta)\sin(\phi)\cos(\phi)\Big) r^2 \sin(\theta) dr d\theta d\phi = 0 \tag{A.44}$$

Next I_{xz}

$$I_{xz} = \rho \int\limits_{\phi=0}^{2\pi} \int\limits_{\theta=0}^{\pi} \int\limits_{r=0}^{R} \Big(- xz\Big) r^2 \sin(\theta) dr d\theta d\phi \tag{A.45}$$

$$I_{xz} = \rho \int\limits_{\phi=0}^{2\pi} \int\limits_{\theta=0}^{\pi} \int\limits_{r=0}^{R} \Big(- r^2 \sin(\theta)\cos(\theta)\cos(\phi)\Big) r^2 \sin(\theta) dr d\theta d\phi = 0 \tag{A.46}$$

Next I_{yz}

$$I_{yz} = \rho \int\limits_{\phi=0}^{2\pi} \int\limits_{\theta=0}^{\pi} \int\limits_{r=0}^{R} \Big(- yz\Big) r^2 \sin(\theta) dr d\theta d\phi \tag{A.47}$$

$$I_{yz} = \rho \int\limits_{\phi=0}^{2\pi} \int\limits_{\theta=0}^{\pi} \int\limits_{r=0}^{R} \Big(- r^2 \sin(\theta)\cos(\theta)\sin(\phi)\Big) r^2 \sin(\theta) dr d\theta d\phi = 0 \tag{A.48}$$

Since I_{pq} is symmetric, then

$$I = MR^2 \begin{bmatrix} \frac{2}{5} & 0 & 0 \\ 0 & \frac{2}{5} & 0 \\ 0 & 0 & \frac{2}{5} \end{bmatrix} \tag{A.49}$$