

**Physics 151**  
**Assignment 5**  
**Due Thursday, November 3, 2005**  
**at the start of class**

Because an overwhelming number of books use the sign convention for the Poisson bracket

$$[A, B] = \sum_j \left( \frac{\partial A}{\partial q_j} \frac{\partial B}{\partial p_j} - \frac{\partial A}{\partial p_j} \frac{\partial B}{\partial q_j} \right) , \quad (1)$$

**we have decided to switch the sign!!! PLEASE USE THIS CONVENTION HERE AND IN FUTURE PROBLEM SETS!**

1. We saw in class that the components of the angular momentum vector  $\vec{L} = \vec{x} \wedge \vec{p}$  for a single particle satisfy the Poisson bracket relations,

$$[L_1, L_2] = L_3 , \quad [L_2, L_3] = L_1 , \quad \text{and} \quad [L_3, L_1] = L_2 . \quad \text{with the new sign!}$$

Show that these are also the correct Poisson bracket relations for the components of the total angular momentum vector for a collection of  $N$  particles,

$$\vec{L} = \sum_{j=1}^N \vec{L}^{(j)} .$$

Here  $\vec{L}^{(j)}$  denotes the angular momentum of the  $j^{\text{th}}$  particle.

2. Let  $L_i$  denote the  $i^{\text{th}}$  component of the vector  $\vec{L}$ . One can define a *scalar* under rotations as quantity  $f(\mathbf{x}, \mathbf{p})$  such that

$$[L_i, f] = 0 , \quad \text{for } i = 1, 2, 3 . \quad (2)$$

One also says that such an  $f$  is *rotationally invariant*.

Also a set of three functions  $\mathbf{f}_1(\mathbf{x}, \mathbf{p})$ ,  $\mathbf{f}_2(\mathbf{x}, \mathbf{p})$ , and  $\mathbf{f}_3(\mathbf{x}, \mathbf{p})$  transform covariantly as a *vector*  $\vec{f}(\mathbf{x}, \mathbf{p})$  under rotations if they satisfy

$$[L_i, \mathbf{f}_j] = \sum_{k=1}^3 \epsilon_{ijk} \mathbf{f}_k . \quad (3)$$

Here  $\epsilon_{123} = 1$  and  $\epsilon_{ijk}$  is totally anti-symmetric in the three indices. Similarly a *rank two tensor* under rotations is a set of nine quantities  $f_{ij}$  such that

$$[L_i, f_{jk}] = \sum_{j', k'} \epsilon_{ijj'} \epsilon_{ikk'} f_{j'k'} . \quad (4)$$

Compute the Poisson brackets to determine whether each of the following expressions are scalars, vectors, rank two tensors, or none of the above. (It is not necessary to compute all the Poisson brackets, if you can recognize patterns in the computation. But you must explain how such patterns can be used to get the desired answer.)

- i.  $f = \vec{x}^2 = x_1^2 + x_2^2 + x_3^2$ .
- ii.  $f = \frac{1}{2m}\vec{p}^2 + \frac{k}{2}\vec{x}^2 + \lambda(\vec{x}^2)^2$ . Here  $m, k, \lambda$  are given constants.
- iii.  $f = \cos(\beta\vec{x} \cdot \vec{p}) = \cos(\beta(x_1p_1 + x_2p_2 + x_3p_3))$ . Here  $\beta$  is a given constant.
- iv.  $f = x_1^3 + x_2^3 + x_3^3$ .
- v. Consider three quantities  $f_j = x_j$ , for  $j = 1, 2, 3$ .
- vi. Consider three quantities  $f_j = (\vec{x} \wedge \vec{p})_j$ .
- vii. In this one define a matrix of nine  $f_{ij}$ 's by  $f_{11} = f_{22} = f_{33} = 0$ ,  $f_{12} = (\vec{x} \wedge \vec{p})_3$ ,  $f_{13} = -(\vec{x} \wedge \vec{p})_2$ ,  $f_{23} = (\vec{x} \wedge \vec{p})_1$ ,  $f_{21} = -f_{12}$ ,  $f_{31} = -f_{13}$ ,  $f_{32} = -f_{23}$ . Explain the relation between this case and the previous case (vi).
- viii. Here one has nine quantities given by  $f_{ij} = x_i p_j$ .
- ix. In this case consider three  $f_j$ 's where  $f_j = x_j \sin(\vec{p}^2)$ .
- x.  $f = e^{-\vec{x}^2 \vec{p}^2}$ .
- xii.  $f_{ij} = \vec{x}^2 \delta_{ij}$ .
- xii. The "Lenz vector"  $\vec{A}$  in a central force problem.
- xiii. In this case, consider a set of  $N$  different vector particles and let  $f_j = \sum_{i=1}^N (\vec{x}^{(i)} \wedge \vec{p}^{(i)})_j$ , for  $j = 1, 2, 3$ .