Integrating Factor Method

A Generalized Method to What We've Seen in Class

Written by Baili Min. Based on my undergraduate text.Generated by LaTeX

We have seen that if a symmetric form

$$P(x,y)dx + Q(x,y)dy = 0 \tag{1}$$

is exact, then we can find $F(x,y)$ such that $dF(x,y) = 0$ and $F(x,y) = C$ solves the equation.

We have also seen that if a symmetric form is not exact, maybe we can multiply a factor to make it exact. Actually when we explored our methods dealing with separable equations, first order linear equations, Bernoulli equations, etc., we were following that idea.

So it is natural to think: for any symmetric form $P(x,y)dx + Q(x,y)dy = 0$, can we find such a non-zero factor which, by multiplying it on both sides, can make the equation an exact form?

This turns out to find non-zero $\mu = \mu(x,y)$ such that

$$\mu(x,y)P(x,y)dx + \mu(x,y)Q(x,y)dy = 0 \tag{2}$$

is exact, i.e.

$$\frac{\partial(\mu P)}{\partial y} = \frac{\partial(\mu Q)}{\partial x}. \tag{3}$$

And if so, we call this $\mu = \mu(x,y)$ our **integrating factor** of the equation.

But the problems are: Does that integrating factor really exist? If it exists, is it easy to get? Actually from thoeries in partial differential equations, this μ does exist but to solve for it turns out to solve our equation (2). Thus in general it is not possible to find solutions to equation (1) by finding the integrating factor from (3).

Luckily, in some cases we might explore more.

For example, if the equation (1) has a integrating factor of only the x variable $\mu = \mu(x)$(like what we did for the first order linear equation!) then by (3) we can easily get

$$Q\frac{d\mu}{dx} = (\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})\mu,$$

or

$$\frac{1}{\mu(x)}\frac{d\mu(x)}{dx} = \frac{1}{Q(x,y)}(\frac{\partial P(x,y)}{\partial y} - \frac{\partial Q(x,y)}{\partial x}) \tag{4}$$

Since LHS(left hand side) is a function of only x, therefore so is RHS(right hand side). Therefore, if our differential equation (1) has an integrating factor $\mu(x)$, we must have(mathematically we say the following is a necessary condition) the expression

$$\frac{1}{Q(x,y)}(\frac{\partial P(x,y)}{\partial y}-\frac{\partial Q(x,y)}{\partial x}) \tag{5}$$

relies only on x, nothing to do with y.

Conversely, if we check that $\frac{1}{Q(x,y)}(\frac{\partial P(x,y)}{\partial y}-\frac{\partial Q(x,y)}{\partial x})=G(x)$ does not rely on y, then we can get one integrating factor

$$\mu(x)=e^{\int G(x)dx} \tag{6}$$

from $\frac{1}{\mu(x)}\frac{d\mu(x)}{dx}=G(x)$.

Therefore we can conclude our theorem:

Theorem 0.1 *The differential equation (1) has an integrating method which relies only on x if and only if the expression (5) relies only on x, which denoted as $G(x)$, and the function defined by (6) is one integrating factor to the differential equation (1).*

Similarly, we can work on the case of y.

Theorem 0.2 *The differential equation (1) has an integrating method which relies only on y if and only if the expression*

$$\frac{1}{P(x,y)}(\frac{\partial Q(x,y)}{\partial x}-\frac{\partial P(x,y)}{\partial y})=H(y)$$

relies only on y and the function defined by $\mu(y)=e^{\int H(y)dy}$ is one integrating factor to the differential equation (1).

Let's see an example.

Example 0.3 *Solve the differential equation*

$$(3x^2+y)dx+(2x^2y-x)dy=0. \tag{7}$$

We check that it is not exact, nor separable, nor first order linear, nor homogeneous. But we see that

$$\frac{1}{Q}(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x})=-\frac{2}{x},$$

which is a function of only x, thus by the theorem (0.1) it has an integrating factor

$$\mu(x) = e^{-\int \frac{2}{x} dx} = \frac{1}{x^2}.$$

Then multiply this $\mu(x)$ on both sides of (7), we get an exact equation

$$3xdx + 2ydy + \frac{ydx - xdy}{x^2} = 0,$$

and finally we get our solution

$$\frac{3}{2}x^2 + y^2 - \frac{y}{x} = C.$$

We can view this from another point. We write (7) into two groups:

$$(3x^3dx + 2x^2ydy) + (ydx - xdy) = 0$$

where the second group $(ydx - xdy)$ obviously has integrating factors $\frac{1}{x^2}$, $\frac{1}{y^2}$ or $\frac{1}{x^2+y^2}$. And consider the first group, we see that $\mu = \frac{1}{x^2}$ is a common integrating factor to both groups. Thus it is an integrating factor to the equation (7).

To make it more general, we need the following theorem. You can try to prove it as an exercise.

Theorem 0.4 *If $\mu = \mu(x, y)$ is an integrating factor to the equation (1) such that*

$$\mu(x, y)P(x, y)dx + \mu(x, y)Q(x, y)dy = dF(x, y),$$

then $\mu(x, y)g(F(x, y))$ is also integrating factor to the equation (1) where g is any non-zero differentiable function.

Now we continue to do this method. If the LHS of the equation (1) can be written into two groups, i.e.

$$(P_1dx + Q_1dy) + (P_2dx + Q_2dy) = 0,$$

where $(P_1dx + Q_1dy)$ has an integrating factor μ_1 and $(P_2dx + Q_2dy)$ has an integrating factor μ_2 such that

$$\mu_1(P_1dx + Q_1dy) = dF_1, \mu_2(P_2dx + Q_2dy) = dF_2.$$

Then from the theorem above, for any non-zero differential functions g_1 and g_2, we know that $\mu_1 g_1(F_1)$ is an integrating factor to the first group while $\mu_2 g_2(F_2)$

is an integrating factor to the second group. So, if we can choose g_1 and g_2 properly such that

$$\mu_1 g_1(F_1) = \mu_2 g_2(F_2),$$

then $\mu = \mu_1 g_1(F_1)$ is an integrating factor to the equation (1).

Let's see another example.

Example 0.5 *Solve the differential equation*

$$(x^3 y - 2y^2)dx + x^4 dy = 0.$$

We group like this:

$$(x^3 y + x^4 dy) - 2y^2 dx = 0.$$

The first group has an integrating factor $\frac{1}{x^3}$ with $xy = C_1$; the second one has an integrating factor $\frac{1}{y^2}$ with $x = C_2$. We need to find g_1 and g_2 such that

$$\frac{1}{x^3} g_1(xy) = \frac{1}{y^2} g_2(x).$$

We can let

$$g_1(xy) = \frac{1}{(xy)^2}, g_2(x) = \frac{1}{x^5}.$$

Then we have an integrating factor

$$\mu = \frac{1}{x^5 y^2}.$$

And we have

$$\frac{1}{(xy)^2} d(xy) - \frac{2}{x^5} dx = 0.$$

Do integration, we can get a general solution

$$y = \frac{2x^3}{Cx^4 + 1},$$

and particular solutions $x = 0$ and $y = 0$, which are missed when we multiply μ.

How about the homogeneous equation $P(x, y)dx + Q(x, y)dy = 0$? It is a good exercise to check that the funciton

$$\mu(x, y) = \frac{1}{xP(x, y) + yQ(x, y)}$$

is an integrating factor.

You may try to use this method to solve homogeneous equations. :)