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Second Exam For Math 600, Fall 2011, Rakesh

pen = final answer, pencil = scratchwork

1. [25] State and prove the Root Test.

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Root Test: Given a^ series $\sum_{n=1}^{\infty} a_n$ (complex terms), for $|a_n|^{\frac{1}{2}}$, if

(i) $|a_n|^{\frac{1}{2}} < 1$, series converges

(ii) $|a_n|^{\frac{1}{2}} > 1$, series diverges

(iii) $|a_n|^{\frac{1}{2}} = 1$, no conclusion

Proof: (i) Given $|a_n|^{\frac{1}{2}} < 1$ and a series $\sum_{n=1}^{\infty} a_n$, then $\sum_{n=1}^{\infty} a_n$ converges if the sequence of partial sums is bounded. So,

$s_1 = a_1$
$s_2 = a_1 + a_2$
$s_3 = a_1 + a_2 + a_3$
$\vdots$

$(|a_1|+|a_2|)^{\frac{1}{2}} \overset{?}{\leq} |a_1|^{\frac{1}{2}} + |a_2|^{\frac{1}{2}}$

$\leq$ Δ inequality

Then $|s_1| = |a_1|$, $|s_2| \leq |a_1| + |a_2|$ and so on. (i) If $|a_n|^{\frac{1}{2}} < 1$, $s_n = a_1 + a_2 + \cdots + a_n + \cdots$

$\Rightarrow |s_n| = |a_1| + |a_2| + \cdots + |a_n| + \cdots$

$\Rightarrow |s_n|^{\frac{1}{2}} = (|a_1| + \cdots |a_n| + \cdots)^{\frac{1}{2}} \leq |a_1|^{\frac{1}{2}} + \cdots + |a_n|^{\frac{1}{2}}$

$|a_3|^{\frac{1}{2}} < 1$

what is $|a_2|^{\frac{1}{2}} \geq |a_3|^{\frac{1}{2}}$?

But for $|x|^{\frac{1}{2}} < 1$ $0 < x < 1$

So if $|a_n|^{\frac{1}{2}} < 1$, $|a_{n-1}|^{\frac{1}{2}} \leq |a_n|^{\frac{1}{2}}$??

$\therefore |a_1|^{\frac{1}{2}} \leq |a_2|^{\frac{1}{2}}$, with all $|a_i|^{\frac{1}{2}} < 1$

$\Rightarrow$ So the sequence $|s_n|^{\frac{1}{2}}$ bdd $\rightarrow$ converge

If $|a_n|^{\frac{1}{2}} > 1$, $|s_n|^{\frac{1}{2}} \leq |a_1|^{\frac{1}{2}} + \cdots + |a_n|^{\frac{1}{2}}$, but the sequence is unbounded $\Rightarrow$ diverge

If $|a_n|^{\frac{1}{2}} = 1$, the series could alternate $\rightarrow$ no conclusion,

$\rightarrow$ since $|a_n| > 1$, every term adds at least 1 to S_{n-1}. $\Rightarrow$ unbounded

$\rightarrow$ $|(-1)^n| = 1$, but S_n is unbounded since it alternates,

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Proposition: Given a complex-valued series, $\sum_{n=1}^{\infty} a_n$, examine $|a_n|$. If

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(i) $|a_n|^{\frac{1}{2}} < 1$, then the series converges.

(ii) $|a_n|^{\frac{1}{2}} > 1$, then the series diverges.

(iii) $|a_n|^{\frac{1}{2}} = 1$, we reach no conclusion.

Proof: Given a complex series $\sum_{n=1}^{\infty} a_n$, we know the series converges if the sequence of partial sums $\{s_n\}$ is bdd. Given ~~s_1~~ a_1, $s_2 = a_1 + a_2, \ldots$, then $|s_1| \leq |a_1|$, $|s_2| \leq |a_1| + |a_2|, \ldots, |s_n| \leq |a_1| + \cdots + |a_n|, \ldots$. By the Triangle Inequality, we know $(|x_1| + \cdots + |x_n|)^{\frac{1}{2}} \leq |x_1|^{\frac{1}{2}} + \cdots + |x_n|^{\frac{1}{2}}$. So, clearly, $|s_n|^{\frac{1}{2}} = (|a_1| + \cdots + |a_n| + \cdots)^{\frac{1}{2}} \leq |a_1|^{\frac{1}{2}} + \cdots + |a_n|^{\frac{1}{2}} + \cdots$. Also, all terms $|a_n|^{\frac{1}{2}}$ are between zero and one (for $|a_n| < 1$). Thus, if $|a_n|^{\frac{1}{2}} < 1$, the sequence is bounded, and converges. If $|a_n|^{\frac{1}{2}} > 1$, then the sequence $\{s_n\}$ is unbounded since $|a_n|^{\frac{1}{2}} > 1$. The series diverges. To show $|a_n|^{\frac{1}{2}} = 1$ is inconclusive, examine an alternating series. There exists a "liminf" and "limsup" not equal. So the series may diverge. Thus, we draw no conclusion.

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