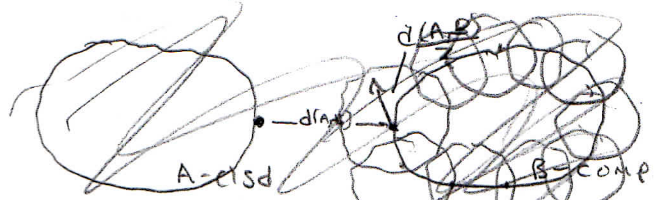
3. [25] Suppose (X, d) is a metric space and A, B are two non-empty **disjoint** subsets of X with A **closed** and B **compact**. Using **sequential compactness**, prove that $d(A, B) > 0$ where

$$d(A, B) = \inf\{d(a, b) \,:\, a \in A,\ b \in B\}.$$



$A \cap B = \phi$

That's open covers!

?? No

Sequential: Let $\{a_n\} \in A$, $\{b_n\} \in B$. Then $d(a_n, b_n)$ converges since $d(A,B)$ exists. Consider $\{b_{n_k}\} \to b_* \in B$.

Since B is compact, $\{b_n\}$ has a convergent subseq. to a point in B. Then $d(a_{n_k}, b_{n_k}) = d(a_{n_k}, b_*)$, but A closed, so it has a limit point $a_* \in A$. So what is $d(a_{n_k}, b_*)$? Well a_* is at least $d(A,B)$ away from b_*, and A, B are disjoint $\Rightarrow$ $\exists N_r(b_*)$ st $\forall N_s(a_{*_i}) \cap N_r(b_*) = \phi$. So choose $r = \frac{d(A,B)}{2}$ and $s = r = \frac{d(A,B)}{2}$. Thus $d(a_*, b_*) > d(A,B)$ since $r > 0$ and $s > 0$ by def of an open nbhd.

NOT HERE

~~inf~~ $\inf\{d(a_n, b_n)\}$

Proof: Let (X,d) be a m.s. and A, B two disjoint non-empty subsets of X where A is closed and B is compact. We show $d(A,B) > 0$ where $d(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$.

Let $\{a_n\} \in A$ and $\{b_n\} \in B$ be sequences. Since B is compact, $\{b_n\}$ has a convergent subsequence to a point in B: $\{b_{n_k}\} \to b_* \in B$. So, $d(a_{n_k}, b_{n_k}) = d(a_{n_k}, b_*)$. Since A is closed and non-empty, it has a limit point $a_* \in A$. Since $A \cap B = \phi$, there exists $N_r(b_*)$, $r > 0$ such that for every $N_s(a_{*_i})$, $a_{*_i} \in A'$, $s > 0$, $N_s(a_{*_i}) \cap N_r(b_*) = \phi$. So choose $r = s = \frac{d(A,B)}{2}$. Thus $d(a_*, b_*) = d(A,B)$, but since r, s are both greater than zero, by definition of $N_s(a_{*_i})$ and $N_r(b_*)$, $d(A,B) > 0$ as well.