From Ronald Meester, *A natural introduction to Probability theory*, page 110.

Definition A function g is said to be *regular* if there exist numbers $\cdots < a_{-1} < a_0 < a_1 < \cdots$ with $a_i \to \infty$ and $a_{-i} \to -\infty$ when $i \to \infty$, so that g is continuous and monotone on each interval (a_i, a_{i+1}).

Theorem Let $X_1, \cdots, X_n$ be independent continuous random variables, and let $g_1, g_2, \cdots, g_n$ be regular functions. Then $g_1(X_1), g_2(X_2), \cdots, g_n(X_n)$ are independent random variables.

Proof Assume for simplicity that $n = 2$. It follows from regularity that for all $x \in \mathbb{R}$ we can write

$$A_1 \equiv \{y : g_1(y) \le x\} = \bigcup_i A_{1,i}(x)$$

and

$$A_2 \equiv \{y : g_2(y) \le x\} = \bigcup_i A_{2,i}(x),$$

as unions of pairwise disjoint intervals. Therefore, we can write

$$\begin{aligned} P(g_1(X_1) \le x, g_2(X_2) \le y) &= \sum_i \sum_j P(X_1 \in A_{1,i}(x), X_2 \in A_{2,j}(y)) \\ &= \sum_i \sum_j P(X_1 \in A_{1,i}(x)) P(X_2 \in A_{2,j}(y)) \\ &= \sum_i P(X_1 \in A_{1,i}(x)) \sum_j P(X_2 \in A_{2,j}(y)) \\ &= P(g_1(X_1) \le x) P(g_2(X_2) \le y) \end{aligned}$$