a) <u>Technical data of the water rocket</u>

- Average initial air pressure= (40±2.5) Psi
- Diameter of the nozzle = (21.0±0.2) mm
- Initial volume of water = (700 ± 5) mL
- Initial volume of air = (1300 ± 5) mL
- Dry weight of water rocket= (300 ± 1) g

Note: The errors of the measured values are the half of the smallest scale increment

<u>Range Of Projectile equation:</u>

$$s = \frac{u^2}{g} sin(2\theta) \tag{12}$$

In this case s is the range of the water rocket in meters, u is the initial velocity of the projectile and pheta is the angle to the horizontal plane

Calculating the initial lift-off velocity

After having derived formula (12) for the range of a projectile (water rocket) we now need to calculate the components of the formula. We know that θ is the launch angle which is measured from the horizontal plane. The constant g is the acceleration due to gravity which we take to be 9.81 ms^{-1}. We now need to find the initial velocity of the rocket.

The launch velocity can be found by multiplying the acceleration multiplied by the amount of time to expend the fuel.

$$U_{rocket} = a * t \tag{13}$$

Where U_{rocket} is the initial velocity of the rocket in ms^{-1}, a is the acceleration of the rocket in ms^{-2} and t is the time for the water to exit the rocket in s. Let's begin by finding the time taken for the water to exit the bottle. The time taken for all the water to leave the rocket depends on the mass of the water (m), and the mass flow rate ($\dot{m}$).

Calculating the average mass flow rate:

The mass flow rate is a measurement of the amount of mass passing through an opening over a period of time. After the launch, the water and the air leave the rocket and due to Newton's second law creates thrust. To calculate the thrust we need to find the mass flow rate for the water and the air. The following equation is used to calculate the mass flow rate:

$$\dot{m} = p * v * A \tag{14}$$

Where $\dot{m}$ is the mass flow rate in kgs^{-1}, p is the density of water in kgm^{-3}, v is the exit velocity in ms^{-1} and A is the area of the nozzle in m^2 which is in this case $0.00035m^2$.

However, since we don't have the exit velocity of the water we will need to use Bernoulli's equation which links the pressure and velocity of fluids. Assuming that the exit velocity is determined by the Bernoulli equation, then this satisfies the equation:

$$\Delta P = \frac{1}{2} * p * v^2 \tag{15}$$

Where ΔP is the average pressure acting on the water in Nm^{-2}. Now by rearranging **Fehler! Verweisquelle konnte nicht gefunden werden.** for the exit velocity (v) we get:

$$v = \sqrt{\frac{2 * \Delta P}{p}} \tag{16}$$

If we now substitute **Fehler! Verweisquelle konnte nicht gefunden werden.** into **Fehler! Verweisquelle konnte nicht gefunden werden.** we get:

$$\dot{\mathrm{m}} = A * \sqrt{2 * p * \Delta P} \tag{17}$$

As water exits the bottle, the pressure inside the bottle is not constant. Thus, we need to calculate the average pressure (ΔP) acting on the water and the air. An equation which can be to find the average pressure is:

$$\Delta P = \frac{P_i + P_f}{2} \tag{18}$$

Where P_i is our initial pressure and P_f is the final pressure.

For water:

We know that our initial pressure is (40±2.5) Psi. We have to find the final pressure using Boyle's law

$$P_i * V_i = P_f * V_f \tag{19}$$

V_i is the initial volume of air which is in this case 1.3 liters and V_f is the final volume which will end up being 2L. Using Boyle's law, we find that the final pressure is 26 Psi. We can now use this value and substitute them into (18) to find that the $\Delta P = 33\ Psi$. Converting this into standard units gives us: 227528.4Nm^{-2}. We can substitute this into (17) to find the $\dot{\mathrm{m}}$ of water:

$$\dot{\mathrm{m}} = 0.000346\mathrm{m} * \sqrt{2 * \ 998 * 227528.4}$$

$$\dot{\mathrm{m}} = 7.37 kgs^{-1}$$

Now that we have calculated $\dot{\mathrm{m}}$ we can determine the time taken for all the water to leave the bottle using the following formula:

$$t = \frac{m}{\dot{\mathrm{m}}} \tag{20}$$

We can solve for time by substituting in our values to find that:

$$t = \frac{0.7}{7.37}$$

$$t = 0.095s$$

For air:

We now need to find the average pressure acting on the remaining air. We have already found that the pressure remaining in the bottle is 26 Psi. We use this as our initial pressure value acting on the air. As final pressure value we take the atmospheric pressure which we take to be 14.7Psi. We substitute these values into (18) to find that $\Delta P = 20.4\ Psi$. Converting this into standard units gives us: 140309.2Nm^{-2}. We can substitute this into (17) to find the ṁ of air:

$$\dot{\mathrm{m}} = 0.000346\mathrm{m} * \sqrt{2 * \ 4.485 * 140309.2}$$

$$\dot{\mathrm{m}} = 0.39kgs^{-1}$$

The mass of air can be determined by multiplying the volume by the density. In this case the mass of air is 0.006kg. Using the mass flow rate of air, we can determine the time taken for all the air to leave the bottle using the following formula:

$$t = \frac{0.006}{0.39}$$

$$t = 0.015s$$

We can now find the time taken for the air and water to be expelled by adding their individual expulsion times.

$$t = 0.015 + 0.095$$

$$t = 0.110s$$

Now let's continue by finding the acceleration of the water rocket upon liftoff. To find the acceleration we use Newton's second law:

$$F_{UN} = m_{av} * a \tag{21}$$

Where F is the unbalanced force the rocket experiences in N, m_{av} is the average mass of the rocket in kg and a is the acceleration of the rocket in ms^{-2}. To solve for acceleration, we need to rearrange (8) to:

$$a = \frac{F_{UN}}{m_{av}} \tag{22}$$

The average mass for this rocket is found by:

$$m_{av} = m_b + \frac{m_w}{2} \tag{23}$$

Where m_w is the mass of the water inside the rocket in kg in this case 0.5kg and m_b is the mass of the bottle in kg in this case 0.1kg. We can use these values to solve for m_{av}:

$$m_{av} = 0.3 + \frac{0.7}{2}$$

$$m_{av} = 0.65kg$$

To find the unbalanced force I considered the four components of flight: lift, thrust, drag and weight. To simplify the equation, I chose to ignore lift and drag considering the small surface area of the water rocket this step is justified. This gives us:

$$F_{UN} = T_f - (m_{av} * g) \tag{24}$$

Where T_f is the thrust experienced by the rocket in N as a result of the water leaving the bottle, T_f can be found by the equation:

$$T_f = \dot{\mathrm{m}} * v \tag{25}$$

Where v is the velocity of the fluid leaving the bottle. The following equation can be used to calculate the exit velocity of a fluid:

$$v = \frac{\dot{\mathrm{m}}}{p * A} \tag{26}$$

https://www.grc.nasa.gov/www/k-12/airplane/thrsteq.html

By substituting in our values of the mass flow rate, the pressure of the fluid and then the area of the nozzle we can find the exit velocity for water (v_w) and air (v_a):

$$v_w = \frac{7.37}{998 * 0.000346}$$

$$v_w = 21.3ms^{-1}$$

$$v_a = \frac{0.39}{4.485 * 0.000346}$$

$$v_a = 251.3ms^{-1}$$

Now we can use v_w and $v_a$to solve (25) for T_f:

$$T_{f\_w} = 7.37 * 21.3$$

$$T_{f\_w} = 157\ N$$

$$T_{f\_a} = 0.39 * 251.3$$

$$T_{f\_a} = 98\ N$$

$$T_f \frac{= (157 + 98)}{2}$$

$$T_t = 127.5\ N$$

Using our value for F_t we can now solve (24) for F_{UN}:

$$F_{UN} = 127.5 - (0.45 * 9.81)$$

$$F_{UN} = 123.1\ N$$

Using F_{UN} and m_{av} we can now solve (22) for the acceleration of the rocket:

$$a = \frac{123.1}{0.65}$$

$$a = 189.4ms^{-2}$$

Finally, we can now use our values t and a to solve (13) for the launch velocity:

$$U_{rocket} = 189.4 * 0.11$$

$$U_{rocket} = 20.8ms^{-1}$$

If we now substitute the value for our initial velocity into (12) we get our range.