The Intermediate Value Theorem

We first prove the following

Lemma 0.1
Let $a < b$ be real numbers. Let $f(x)$ be a continuous function, $f : [a,b] \mapsto \mathbb{R}$.
If $x_0 \in [a,b]$ and $f(x_0) > 0$ then there exists a $\delta > 0$ such that for all $x \in (x_0 - \delta, x_0 + \delta) \cap [a,b]$ one has: $f(x) > 0$.
If $x_0 \in [a,b]$ and $f(x_0) < 0$ then there exists a $\delta > 0$ such that for all $x \in (x_0 - \delta, x_0 + \delta) \cap [a,b]$ one has: $f(x) < 0$.

Proof. f is a continuous function. Hence for any $\varepsilon > 0$ there is $\delta > 0$ s. t. for any $x \in (x_0 - \delta, x_0 + \delta) \cap [a,b]$

$$|f(x) - f(x_0)| < \varepsilon \quad \Leftrightarrow \quad f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon. \tag{1}$$

If $f(x_0) > 0$ then put $\varepsilon = \frac{1}{2} f(x_0) > 0$. Inequalities (1) imply that for the corresponding $\delta > 0$ and any $x \in (x_0 - \delta, x_0 + \delta) \cap [a,b]$

$$f(x) > f(x_0) - \varepsilon = \frac{1}{2} f(x_0) > 0.$$

If $f(x_0) < 0$ then put $\varepsilon = -\frac{1}{2} f(x_0) >$. Inequalities (1) imply that for the corresponding $\delta > 0$ and any $x \in (x_0 - \delta, x_0 + \delta) \cap [a,b]$

$$f(x) < f(x_0) + \varepsilon = \frac{1}{2} f(x_0) < 0.$$

□
The following statement is called *the Intermediate Value Theorem.*

Theorem 0.2
Let $a < b$ be real numbers. Let $f(x)$ be a continuous function, $f : [a,b] \mapsto \mathbb{R}$. If $f(a) \le 0$ and $f(b) \ge 0$ then there exists a $c \in [a,b]$ such that $f(c) = 0$.

Proof. Consider the set $X \stackrel{\text{def}}{=} \{x \in [a,b]$ and such that $f(x) \le 0\}$. Note that $X \ne \emptyset$ because $a \in X$ and it is bounded because $X \subset [a,b]$. Hence, according to a previously proved theorem, it has the lowest upper bound (or supremum); put $x_0 \stackrel{\text{def}}{=} \sup X$.
There are now three options: (i) $f(x_0) < 0$, or (ii) $f(x_0) > 0$, or (iii) $f(x_0) = 0$. We shall first show that (i) and (ii) are impossible.
Suppose that $f(x_0)) < 0$. Note that in this case $x_0 < b$ because $f(b) \ge 0$. By the Lemma, there is a $\delta > 0$ such that $f(x) < 0$ for any $x \in (x_0 - \delta, x_0 + \delta)$. But then $\sup X \ge x_0 + \delta > x_0$ because every $x \in [x_0, x_0 + \delta)$ belongs to X. This contradiction implies that $f(x_0)) \not< 0$.
Suppose that $f(x_0)) > 0$. Note that in this case $x_0 > a$ because $f(a) \le 0$. By the Lemma, there is a $\delta > 0$ such that $f(x) > 0$ for any $x \in (x_0 - \delta, x_0 + \delta)$. But then $\sup X \le x_0 - \delta < x_0$ because if $x \in (x_0 - \delta, x_0]$ then this $x \notin X$. This contradiction implies that $f(x_0)) \not> 0$.
Hence $f(x_0) = 0$ and we can put $c = x_0$. □