only to vectors Φ such that

$$\sum_n n \, \|\varphi^{(n)}\|^2 < \infty \, . \tag{36}$$

By construction, Fock space is populated by repeated application of creation operators on the vacuum vector:

$$0 \oplus \cdots \oplus 0 \oplus \varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n \oplus 0 \oplus \cdots = a^\dagger(\varphi_1) a^\dagger(\varphi_2) \cdots a^\dagger(\varphi_n) \, \Omega \, . \tag{37}$$

A bosonic creation operator is always unbounded while one can show that the norm of a fermionic creation operator $a^\dagger(\varphi)$ is equal to $\|\varphi\|$.

3.2 Product Fock spaces

A very important property of both bosonic and fermionic Fock space is their behaviour with respect to the one-particle space: there is a natural isomorphism, i.e., a unitary map

$$U : \Gamma^{(\mathrm{a})\mathrm{s}}(\mathcal{H}_1 \oplus \mathcal{H}_2) \to \Gamma^{(\mathrm{a})\mathrm{s}}(\mathcal{H}_1) \otimes \Gamma^{(\mathrm{a})\mathrm{s}}(\mathcal{H}_2) \, . \tag{38}$$

U is actually a direct sum of unitary maps U_n between the n-particle space of $\Gamma^{(\mathrm{a})\mathrm{s}}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ and all the subspaces of $\Gamma^{(\mathrm{a})\mathrm{s}}(\mathcal{H}_1) \otimes \Gamma^{(\mathrm{a})\mathrm{s}}(\mathcal{H}_2)$ that contain k particles in the first factor and $n-k$ in the second, $k = 0, 1, \ldots, n$:

$$U_n : (\mathcal{H}_1 \oplus \mathcal{H}_2)^{\otimes n, (\mathrm{a})\mathrm{s}} \to \bigoplus_{k=0}^{n} \Big(\mathcal{H}_1^{\otimes k, (\mathrm{a})\mathrm{s}} \otimes \mathcal{H}_2^{\otimes n-k, (\mathrm{a})\mathrm{s}} \Big) \, . \tag{39}$$

The explicit form differs a bit for bosons and fermions. For bosons

$$\begin{aligned}
& U_1(\varphi_1 \oplus \varphi_2) = \varphi_1 \oplus \varphi_2 \\
& U_2\big((\varphi_1 \oplus \varphi_2) \odot (\psi_1 \oplus \psi_2)\big) = (\varphi_1 \odot \psi_1) \oplus (\varphi_1 \otimes \psi_2 + \psi_1 \otimes \varphi_2) \\
& \quad \oplus (\varphi_2 \odot \psi_2) \\
& \vdots
\end{aligned} \tag{40}$$

while for fermions

$$\begin{aligned}
& U_1(\varphi_1 \oplus \varphi_2) = \varphi_1 \oplus \varphi_2 \\
& U_2\big((\varphi_1 \oplus \varphi_2) \wedge (\psi_1 \oplus \psi_2)\big) = (\varphi_1 \wedge \psi_1) \oplus (\varphi_1 \otimes \psi_2 - \psi_1 \otimes \varphi_2) \\
& \quad \oplus (\varphi_2 \wedge \psi_2) \\
& \vdots
\end{aligned} \tag{41}$$

Suppose that we have a composite system consisting of n indistinguishable parties each with Hilbert space $\mathcal{H}$. For very low densities one could restrict attention to a single particle version of the system. We would then use the Hilbert space

$$\underbrace{\mathcal{H} \oplus \mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{n\text{-times}} \text{ instead of } \mathcal{H}^{\otimes n,\text{(a)s}}\,. \tag{42}$$

For two particles the right choice would be

$$\Big(\bigoplus_i \mathcal{H}_i^{\otimes 2,\text{(a)s}}\Big) \oplus \Big(\bigoplus_{\substack{i_1,i_2\\ i_1<i_2}} \mathcal{H}_{i_1} \otimes \mathcal{H}_{i_2}\Big)\,. \tag{43}$$

There is a simple global prescription for this unitary isomorphism $U = \oplus_n U_n$ in terms of the creation operators

$$U\,\Omega_{12} = \Omega_1 \otimes \Omega_2 \tag{44}$$

$$U\,a^\dagger(\varphi_1 \oplus \varphi_2)\,U^\dagger = a^\dagger(\varphi_1) \otimes \mathbb{1} + \mathbb{1} \otimes a^\dagger(\varphi_2) \text{ for bosons} \tag{45}$$

$$U\,a^\dagger(\varphi_1 \oplus \varphi_2)\,U^\dagger = a^\dagger(\varphi_1) \otimes \mathbb{1} + (-\mathbb{1})^N \otimes a^\dagger(\varphi_2) \text{ for fermions.} \tag{46}$$

Here, Ω_1, Ω_2, and Ω_{12} are the Fock vacua for the one-particle spaces $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_1 \oplus \mathcal{H}_2$.

The isomorphism (38) is essential for resolving the Gibbs paradox for systems with indistinguishable particles. If we consider two large volumes Λ_1 and Λ_2 in $\mathbb{R}^3$, then an equilibrium state of the total volume $\Lambda_1 \cup \Lambda_2$ with $N_1 + N_2$ particles will, up to boundary contributions, correspond with the tensor product of the equilibrium state with N_1 particles in Λ_1 and N_2 in Λ_2., where $N_1 = \rho|\Lambda_1|$ and $N_2 = \rho|\Lambda_2|$, ρ being the particle density. Quantities like entropy or internal energy will then be essentially additive, as is expected for extensive quantities.

Another application of the isomorphism is to construct Fock space $\Gamma^{\text{(a)s}}(\mathcal{H})$ using single mode factors $\Gamma^{\text{(a)s}}(\mathbb{C})$. These single mode spaces are very common.

For fermions $\Gamma^{\text{as}}(\mathbb{C}) \sim \mathbb{C}^2$ with

$$\Omega \sim \begin{pmatrix}1\\0\end{pmatrix} \text{ and } a^\dagger \sim \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\,.$$

The isomorphism U between $\Gamma^{\mathrm{as}}(\mathbb{C}^n)$ and $\mathbb{C}^{2^{\otimes n}}$ is called the Jordan-Wigner isomorphism. Using the standard basis of $\mathbb{C}^2$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{47}$$

we obtain

$$U a_{i_1}^\dagger a_{i_2}^\dagger \cdots a_{i_m}^\dagger \Omega = |\mathbf{i}\rangle, \;\; i_1 < i_2 < \cdots < i_m \,. \tag{48}$$

Here $|\mathbf{i}\rangle$ is the binary string of length m consisting of a 1 at sites $i_1, i_2, \ldots, i_m$ and 0 elsewhere. The Jordan-Wigner transform of a creation operator is

$$U a_i^\dagger U^\dagger = \sigma^z \otimes \cdots \otimes \sigma^z \otimes a^\dagger \otimes \mathbb{1} \cdots \otimes \mathbb{1} \,. \tag{49}$$

The matrix $a^\dagger$ sits at the i-th place and

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = (-\mathbb{1})^{a^\dagger a} \,. \tag{50}$$

For bosons $\Gamma^{\mathrm{s}}(\mathbb{C}) = \mathcal{L}^2(\mathbb{R}, dx)$ and we use the coordinate representation of position and momentum to define $a^\dagger$:

$$(Q\,\psi)(x) = x\psi(x) \text{ and } (P\,\psi)(x) = -\mathrm{i}\psi'(x) \,. \tag{51}$$

We then have

$$\Omega(x) \sim \frac{1}{\pi^{1/4}}\,\mathrm{e}^{-x^2/2} \text{ and } a^\dagger \sim \frac{1}{\sqrt{2}}\,(Q - \mathrm{i}P) \,. \tag{52}$$

The Fock space of a one-particle space $\mathcal{H}$ decomposes into a tensor product of simple harmonic oscillator subspaces.

The exponential vectors are quite useful for bosons

$$\mathrm{Exp}(\varphi) = 1 \oplus \varphi \oplus \frac{1}{\sqrt{2!}}\,\varphi \otimes \varphi \,\oplus \ldots \tag{53}$$

$$= \Omega + a^\dagger(\varphi)\Omega + \frac{1}{2!}\,a^\dagger(\varphi)^2\Omega + \cdots \tag{54}$$

$$= \exp(a^\dagger(\varphi))\,\Omega \,. \tag{55}$$

Exponential vectors are linearly independent: the only possibility for having

$$\sum_{j=1}^{n} \alpha_j \mathrm{Exp}(\varphi_j) = 0 \text{ with } i \neq j \Longrightarrow \varphi_i \neq \varphi_j \tag{56}$$