subgroup described by the definite article "the" is characteristic. In Problem 2.7, for example, we shall define the "Frattini subgroup" of a group. Without referring to the definition, the reader should understand that the Frattini subgroup of any group is characteristic.

An important example of an automorphism of G is the *inner automorphism* θ_g induced by an element $g \in G$. This is the map

$$\theta_g(x) = g^{-1}xg.$$

(The reader should check that θ_g is really an automorphism.) A fairly standard notation that we shall adopt is

$$x^g = g^{-1}xg$$

for $x, g \in G$. The element x^g is said to be the *conjugate* of x with respect to g. In this language, the inner automorphism induced by g is the corresponding conjugation map. Observe that if x and g commute, then $x^g = x$, and thus in an abelian group, inner automorphisms are trivial. (As if to compensate for this, another type of automorphism exists only in abelian groups: this is the map $\theta(x) = x^{-1}$ for $x \in G$.)

The set $\mathrm{Aut}(G)$ of all automorphisms of G is a subgroup of $\mathrm{Sym}(G)$, and the set $\mathrm{Inn}(G)$ of inner automorphisms is a subgroup of $\mathrm{Aut}(G)$. (The reader should check these assertions.)

Let us go back to the situation of an isomorphism $\theta : G_1 \to G_2$. It should be clear that if $H \subseteq G_1$ is a subgroup, then $\theta(H)$ is a subgroup of G_2. In particular, automorphisms map subgroups to subgroups. The subgroup

$$H^g = \{h^g \mid h \in H\}$$

is a subgroup *conjugate* to H. It is, of course, the image of H under the inner automorphism induced by g.

Since characteristic subgroups are fixed by all automorphisms, they are surely fixed by inner automorphisms, and so if C char G, then $C = C^g$ for all $g \in G$. (Note that this is completely obvious in the case $C = \mathbf{Z}(G)$, since then $x^g = x$ for all $x \in C$. In general, the equation $C^g = C$ does not imply that $x^g = x$ for all $x \in C$.)

This leads us to the definition of what is certainly one of the most important concepts in group theory.

(2.14) DEFINITION. A subgroup $N \subseteq G$ is *normal* if $N^g = N$ for all $g \in G$. We write $N \triangleleft G$ in this situation.

In other words, the normal subgroups of a group are precisely those subgroups fixed by all inner automorphisms. All characteristic subgroups are normal and all subgroups of abelian groups are normal. Of course, the subgroups 1 and G are always normal in any group G.

(2.15) LEMMA. *Let $H \subseteq G$ be a subgroup. Then, $H \triangleleft G$ if $H^g \subseteq H$ for all $g \in G$.*