CHAPTER ELEVEN

Module Theory without Rings

11A

In Chapter 12 we will begin our study of rings and their associated modules. As will be apparent then, module theory is a crucial part of ring theory, and that makes the title of this chapter somewhat paradoxical. If R is a ring (whatever that is), then an R-module M is an abelian group for which R is a set of operators. It is more than that, of course, since we make assumptions that relate the action of R on M to the internal structure of R. (This is analogous to the situation of a group acting on a set, where the permutations effected on the set by the group elements are not arbitrary, but instead reflect the structure of the group.) Some of the basic definitions and facts in module theory, however, ignore the structure of the ring and are really concerned only with the properties of the module as an abelian R-group. It is this part of module theory with which we deal in this chapter. (Actually, much of what we do can be made to work in the nonabelian case too, but because our real interest is module theory, we do not pursue that.)

To avoid confusion, we do not refer to the objects of study here as "modules," since that word is usually reserved for the situation where we have a ring and additional structure. Our attention is directed mostly to abelian X-groups, which we usually write additively, so that the symbol "0" means either the identity element or the trivial X-subgroup. For additively written groups, it is customary to write the action of X multiplicatively rather than exponentially. In place of u^x, therefore, we write ux, where $x \in X$ and u lies in some (abelian) X-group M. We thus have $(u + v)x = ux + vx$ for $x \in X$ and $u, v \in M$.

11B

It would be too restrictive to assume that our X-groups are finite, but from time to time, we need conditions that will guarantee that they are not "too infinite." In other words, we wish to investigate useful "finiteness conditions" on abelian X-groups. We have already seen one such condition: finite composition length. We also discuss finite generation and the ascending and descending "chain conditions." It turns out that there are a number of interconnections among these four finiteness conditions.

In order to discuss the two chain conditions, it is convenient to consider a more general situation: that of "partially ordered sets." Although this material is arguably set theory rather than algebra, it is so useful for our purposes that it seems appropriate to include it.

(11.1) DEFINITION. Let P be a set and let $\leq$ be a binary relation on P. Then P is a *partially ordered set*, or *poset*, with respect to $\leq$ provided

i. $a \leq a$ for all $a \in P$,
ii. if $a \leq b$ and $b \leq c$, then $a \leq c$, and
iii. if $a \leq b$ and $b \leq a$, then $a = b$.

Since very little is required in Definition 11.1, examples of posets abound. For instance, if S is any set, then the collection of all subsets of S is a poset with respect to set containment. Also, if G is any group, then the collections of all subgroups and of all normal subgroups form posets, also with respect to containment. Given any poset, one can "reverse the inequality" to construct a new poset, which is called the *dual* of the original poset. (In the dual of P, we have $a \leq b$ iff $b \leq a$ in P.)

Given $a \neq b$ in poset P, it may be that a and b are not comparable, that neither $a \leq b$ nor $b \leq a$. In some posets, every two elements are comparable. If this happens, the set is said to be *linearly* or *totally* ordered. Examples of this are the real numbers and the natural numbers with respect to the ordinary inequality $\leq$. An "algebra" example is the collection of all subgroups of a cyclic p-group with respect to $\subseteq$.

An *ascending chain* in a poset P is an infinite list of not necessarily distinct elements $a_1, a_2, a_3, \ldots$ of P, subscripted by the natural numbers, and such that

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots .$$

Similarly, a *descending chain* in P is such a list with

$$a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots .$$

(Here, as is customary, we write $x \geq y$ to mean $y \leq x$. We also write $x > y$ or $y < x$ to mean $y \leq x$ but $y \neq x$.)

We say that P satisfies the *ACC*, or *ascending chain condition*, if every ascending chain in P is "eventually constant." This means that if $a_1 \leq a_2 \leq \cdots$ in P, then for some integer n, we have $a_n = a_{n+1} = a_{n+2} = \cdots$. (In general, the number n depends on the particular ascending chain considered.) In other words, there are at most finitely many strict inequalities between consecutive terms in every ascending chain. Similarly, the poset P satisfies the *DCC*, or *descending chain condition*, if every descending chain is eventually constant. Obviously, the poset P satisfies the DCC iff its dual satisfies the ACC.

Some examples are appropriate. If P is finite, it satisfies both the ACC and the DCC. Also, if P is any set, we can make P into a poset that satisfies both chain conditions simply by defining $\le$ so that $a \le b$ iff $a = b$. More interesting are the examples

$$P = \{1/n \mid n \in \mathbb{Z}, n > 0\}$$

and

$$Q = \{1 - 1/n \mid n \in \mathbb{Z}, n > 0\}.$$

It should be clear that P and Q are linearly ordered posets with respect to ordinary inequality and that P satisfies the ACC but not the DCC and Q satisfies the DCC but not the ACC.

We give one more example. Let $\mathbb{Z}$ be the additive group of the integers and let P be the poset of all subgroups of $\mathbb{Z}$. Every nontrivial subgroup of $\mathbb{Z}$ has finite index, and so if $H_1 \subseteq H_2 \subseteq \cdots$ is any ascending chain containing some nontrivial subgroup H_k, then there are only finitely many different subgroups H_m with $m \ge k$, and so the chain is eventually constant. If the given chain has no nontrivial member, then, of course, $0 = H_1 = H_2 = H_3 = \cdots$, and in this case too, the chain is eventually constant. It follows that P satisfies the ACC. It does not, however, satisfy the DCC since, for instance,

$$\langle 2 \rangle > \langle 4 \rangle > \langle 8 \rangle > \langle 16 \rangle > \cdots.$$

Before we return to algebra, we need to consider two more conditions on a poset P: the "maximal" and "minimal" conditions. If $S \subseteq P$, we say that $a \in S$ is a *maximal* element of S if there is no element $b \in S$ with $b > a$. Similarly, $a \in S$ is *minimal* if there is no $b \in S$ with $b < a$. Of course, if S is nonempty but finite, it necessarily contains both maximal and minimal elements. (There may be more than one of each, and some elements of S may be both maximal and minimal.) In general, however, nonempty subsets of a poset may fail to have maximal or minimal elements.

The poset P satisfies the *maximal condition* if every nonempty subset has a maximal element and, dually, it satisfies the *minimal condition* if every nonempty subset has a minimal element. Of course, finite posets satisfy both the maximal and minimal conditions.

(11.2) LEMMA. *Let P be any poset. Then*

a. *P satisfies the ACC iff it satisfies the maximal condition and*
b. *P satisfies the DCC iff it satisfies the minimal condition.*

To prove Lemma 11.2 we need to venture more deeply into the realm of set theory and discuss the axiom of choice (which we accept as "true" and use with only the slightest hesitation).

Let $\mathcal{S}$ be a collection of nonempty sets. If $X \in \mathcal{S}$, we can certainly choose an element $x \in X$. This, of course, does not require any mysterious axiom; it follows from the definition of what it means to say that X is "nonempty." Now suppose we wish to build a "machine" to do the job for us and select $x \in X$ whenever $X \in \mathcal{S}$.

This may be an easy task. For instance, if each X consists of only some natural numbers, we can program our machine always to choose the smallest $x \in X$. This "one line program" will do the job. As another example of where we could build our machine, consider the case where the collection $\mathcal{S}$ of sets is finite. We can go through all $X \in \mathcal{S}$ and make a selection in advance of $x \in X$ for each X. We can now explicitly design our machine to return our preselected choice when it is presented with any $X \in \mathcal{S}$.

In the case where $\mathcal{S}$ is infinite, we cannot, in general, go through all $X \in \mathcal{S}$ and for each one make an explicit choice of $x \in X$ (although, as we have seen, this can sometimes be easy, depending on the particular collection of sets). In essence, the axiom of choice says that a "choice machine" can always be built. In other words, there exists a *choice function* $f : \mathcal{S} \to \bigcup \mathcal{S}$ such that $f(X) \in X$ for all $X \in \mathcal{S}$. The axiom tells us only that this functions exists; it gives no clue as to its construction.

We stress that we have been using the word "machine" metaphorically here. The axiom of choice definitely does not guarantee the existence of an actual algorithm that could be programmed on a real computer. On the contrary, the axiom of choice is useful precisely when no such algorithm exists. It allows us to assume the existence of a choice function even when there is no choice algorithm.

We illustrate the situation with this example: Given infinitely many pairs of shoes, an easy rule that serves to pick one shoe from each pair is to choose the right shoe. If we are given infinitely many pairs of socks, however, the axiom of choice is needed to guarantee the existence of a rule to pick one from each pair, but it does not tell us how to do it.

Proof of Lemma 11.2. Part (b) follows from (a) if we replace P by its dual poset, and so we prove only (a). If P satisfies the maximal condition and $a_1 \leq a_2 \leq \cdots$ is any ascending chain, let $S = \{a_i \mid i \geq 1\}$. Since $S \neq \varnothing$, the maximal condition guarantees the existence of a maximal element $a \in S$ and we can write $a = a_n$ for some n. Since $a_m \geq a_n$ for $m \geq n$, we have $a_m = a_n$ for these m, and the chain is eventually constant and P satisfies the ACC.

Conversely, assume P satisfies the ACC and let $S \subseteq P$ be nonempty. Suppose S has no maximal element. Then for every $a \in S$, the set

$$S(a) = \{b \in S \mid b > a\}$$

is nonempty. By the axiom of choice, there exists a choice function that selects some element $b \in S(a)$ for every $a \in S$. In other words, there exists $f : S \to S$ such that $f(a) > a$ for every $a \in S$.

Now choose $a \in S$ and define $a_n \in S$ inductively by setting $a_1 = a$, $a_2 = f(a_1)$, and in general, for $n > 1$, let $a_n = f(a_{n-1})$. Then $a_1 \leq a_2 \leq \cdots$ is an ascending chain in P in which no two consecutive terms are equal. This contradicts the ACC and completes the proof. ∎

Finally we return to algebra. Let M be an abelian X-group and consider the poset of all X-subgroups of M ordered by inclusion: $\subseteq$. We say that M is *noetherian* (after Emmy Noether) if this poset satisfies the ACC. Also, M is *artinian* (after Emil

Artin) if it satisfies the DCC. (Often, and somewhat improperly, one says that M satisfies the ACC or DCC if it is noetherian or artinian.)

Of course, if M is finite, it is both noetherian and artinian. Taking $X = \varnothing$, we see that $\mathbb{Z}$, the additive group of the integers, is noetherian but not artinian. For an example of an artinian but not noetherian group (with $X = \varnothing$), fix a prime number p and consider the multiplicative subgroup of the complex numbers consisting of all elements with order a power of p.

A finiteness condition we have already studied, finite composition length, can be viewed as "decomposing" into the two chain conditions. This is the content of our next result.

(11.3) THEOREM. *Let M be an abelian X-group. Then M has finite composition length iff M is both noetherian and artinian.*

Proof. Suppose M is both noetherian and artinian. Let $\mathcal{S}$ be the set consisting of those X-subgroups of M that have finite composition length, and note that $\mathcal{S} \neq \varnothing$ since $0 \in \mathcal{S}$. Our goal is to show that $M \in \mathcal{S}$.

Since M is noetherian, the poset of all X-subgroups of M satisfies the maximal condition by Lemma 11.2, and hence there exists a maximal element S in $\mathcal{S}$. We may assume that $S < M$ and we let $\mathcal{T}$ be the collection of all X-subgroups of M that properly contain S. Since $M \in \mathcal{T}$, we see that $\mathcal{T}$ is not the empty set, and since M is artinian, Lemma 11.2 guarantees the existence of some minimal element $T \in \mathcal{T}$.

Now $T > S$, but there does not exist any X-subgroup U with $T > U > S$, because otherwise $U \in \mathcal{T}$ and this would contradict the minimality of T. We conclude, therefore, that T/S is X-simple, and so we can append T to the end of an X-composition series for S and get an X-composition series for T. Therefore $\ell(T) < \infty$ and hence $T \in \mathcal{S}$. Since $T > S$, this contradicts the maximality of S.

Conversely, assume that M has finite length. If $S_1 \subseteq S_2 \subseteq \cdots$ is any ascending chain of S-subgroups, then since M is abelian, each $S_i \lhd M$ and so

$$\ell(S_1) \leq \ell(S_2) \leq \cdots \leq \ell(M) < \infty$$

by Lemma 10.7. It follows that the sequence $\{\ell(S_i)\}$ of integers is eventually constant, and for some integer $n \geq 1$ we have

$$\ell(S_n) = \ell(S_{n+1}) = \cdots .$$

By Corollary 10.8, this forces $S_n = S_{n+1} = \cdots$, and hence M is noetherian.

If, on the other hand, $S_1 \supseteq S_2 \supseteq \cdots$ is any descending chain of X-subgroups, then by Lemma 10.7,

$$\ell(S_1) \geq \ell(S_2) \geq \cdots ,$$

and the sequence of integers $\{\ell(S_i)\}$ must eventually be constant in this case, too. Reasoning as above and using Corollary 10.8, we have $S_n = S_{n+1} = \cdots$ for some n, and therefore M is artinian. ■