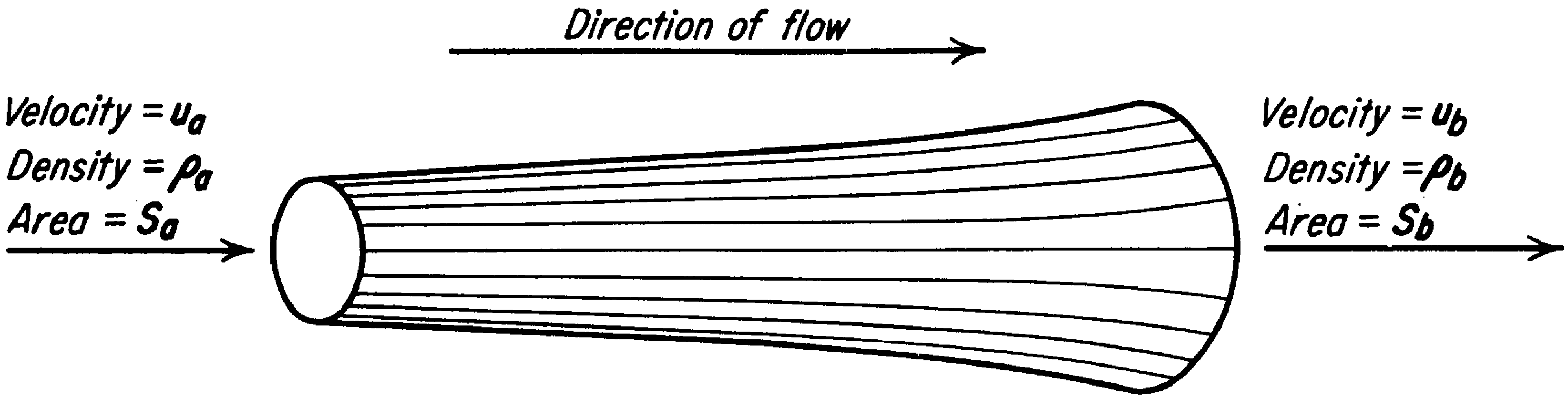
Isentropic Flow of Inviscid Fluid



$$Q = 0 \quad \Delta S = 0$$

In this case The mass balance and MEB are the same as that for the isothermal case.

Now though the total energy balance will give a relation between the velocity and temperature

Total Energy Balance

$$\dot{m}\left[\underbrace{\cancel{\alpha}}_{1} u\frac{du}{dx} + g\underbrace{\cancel{\frac{dZ}{dx}}}_{\text{horizontal}} + C_p\frac{dT}{dx}\right] - \underbrace{\cancel{\frac{dQ}{dx}}}_{\text{adiabatic}} = 0$$

$$\boxed{u\frac{du}{dx} + C_p\frac{dT}{dx} = 0}$$

Equation of State

$$\frac{1}{p}\frac{dp}{dx}-\frac{1}{\rho}\frac{d\rho}{dx}-\frac{1}{T}\frac{dT}{dx}=0$$

Given the normal equation of state, the TEB, MEB, and the thermodynamic relation $C_p - C_v = zR/M$, isentropic flow gives the following useful values.

Useful Relationships

Given the normal equation of state, the TEB, MEB, and the thermodynamic relation $C_p - C_v = zR/M$, isentropic flow gives the following useful values.

$$pV^{\gamma} = p_0 V_0^{\gamma}$$

$$\frac{p}{\rho^{\gamma}} = \frac{p_0}{\rho_0^{\gamma}} \qquad \gamma = \frac{C_p}{C_v}$$

$$\frac{p}{p_0} = \left(\frac{T}{T_0}\right)^{\gamma/(\gamma-1)}$$

From Mechanical Energy Balance

$$u\frac{du}{dx}+\frac{1}{\rho}\frac{dp}{dx}=0 \qquad \text{or} \qquad udu+\frac{1}{\rho}dp=0$$

$$udu=-\frac{1}{\rho}dp=-\left[\rho_0\left(\frac{p}{p_0}\right)^{1/\gamma}\right]^{-1}dp$$

Integrating

$$u^2-u_0^2=\frac{2p_0\gamma}{\rho_0(\gamma-1)}\left[1-\left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\right] \longleftarrow \text{u} \leftrightarrow \text{p}$$

Isentropic Flow

$$u^2 - u_0^2 = \frac{2zRT_0}{M}\frac{\gamma}{(\gamma - 1)}\left[1 - \left(\frac{T}{T_0}\right)\right] \quad \longleftarrow \text{u} \leftrightarrow \text{T}$$

$$u^2 - u_0^2 = \frac{2p_0\gamma}{\rho_0(\gamma - 1)}\left[1 - \left(\frac{\rho}{\rho_0}\right)^{\gamma - 1}\right] \quad \longleftarrow \text{u} \leftrightarrow \rho$$

Velocity, N_{Ma}, and Stagnation

For isentropic flow the definition of the speed of sound is:

$$u_{S,S} \equiv \sqrt{\left(\frac{dp}{d\rho}\right)_S} = \sqrt{\frac{\gamma\, p}{\rho}} = \sqrt{\frac{\gamma RT}{M}}$$

It is also convenient to express the relationships in terms of a reference state where $u_0 = 0$. This is called the stagnation condition ($u_0 = 0$) and P_0 and T_0 are the stagnation pressure and temperature.

Velocity – Mach Relationships

The previous relationships now become:

$$N_{Ma}^2 = \frac{2}{\gamma - 1}\left[\left(\frac{p_0}{p}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]$$

and

$$N_{Ma}^2 = \frac{2}{\gamma - 1}\left[\left(\frac{T_0}{T}\right) - 1\right]$$

Cross-Sectional Area for Sonic Flow

Application of the continuity (mass balance) equation gives:

$$\frac{S}{S^*} = \frac{1}{N_{Ma}}\left[\frac{2+(\gamma-1)N_{Ma}^2}{\gamma+1}\right]^{(\gamma+1)/2(\gamma-1)}$$

S^* is a useful quantity. It is the cross-sectional area that would give sonic velocity (N_{Ma} = 1).

Summary of Equations for Isentropic Flow

$$\frac{S}{S^*} = \frac{1}{N_{Ma}}\left[\frac{2+(\gamma-1)N_{Ma}^2}{\gamma+1}\right]^{(\gamma+1)/2(\gamma-1)}$$

$$\frac{p}{p_0} = \left[1+\frac{(\gamma-1)}{2}N_{Ma}^2\right]^{\gamma/(1-\gamma)}$$

$$\frac{T}{T_0} = \left[1+\frac{(\gamma-1)}{2}N_{Ma}^2\right]^{-1}$$

$$\frac{\rho}{\rho_0} = \left[1+\frac{(\gamma-1)}{2}N_{Ma}^2\right]^{1/(1-\gamma)}$$

p_0, T_0, ρ_0, are at the stagnant (reservoir) conditions.

These ratios are often tabulated versus N_{Ma} for air (γ = 1.4). One must use the equations for gases with $\gamma \neq 1.4$.

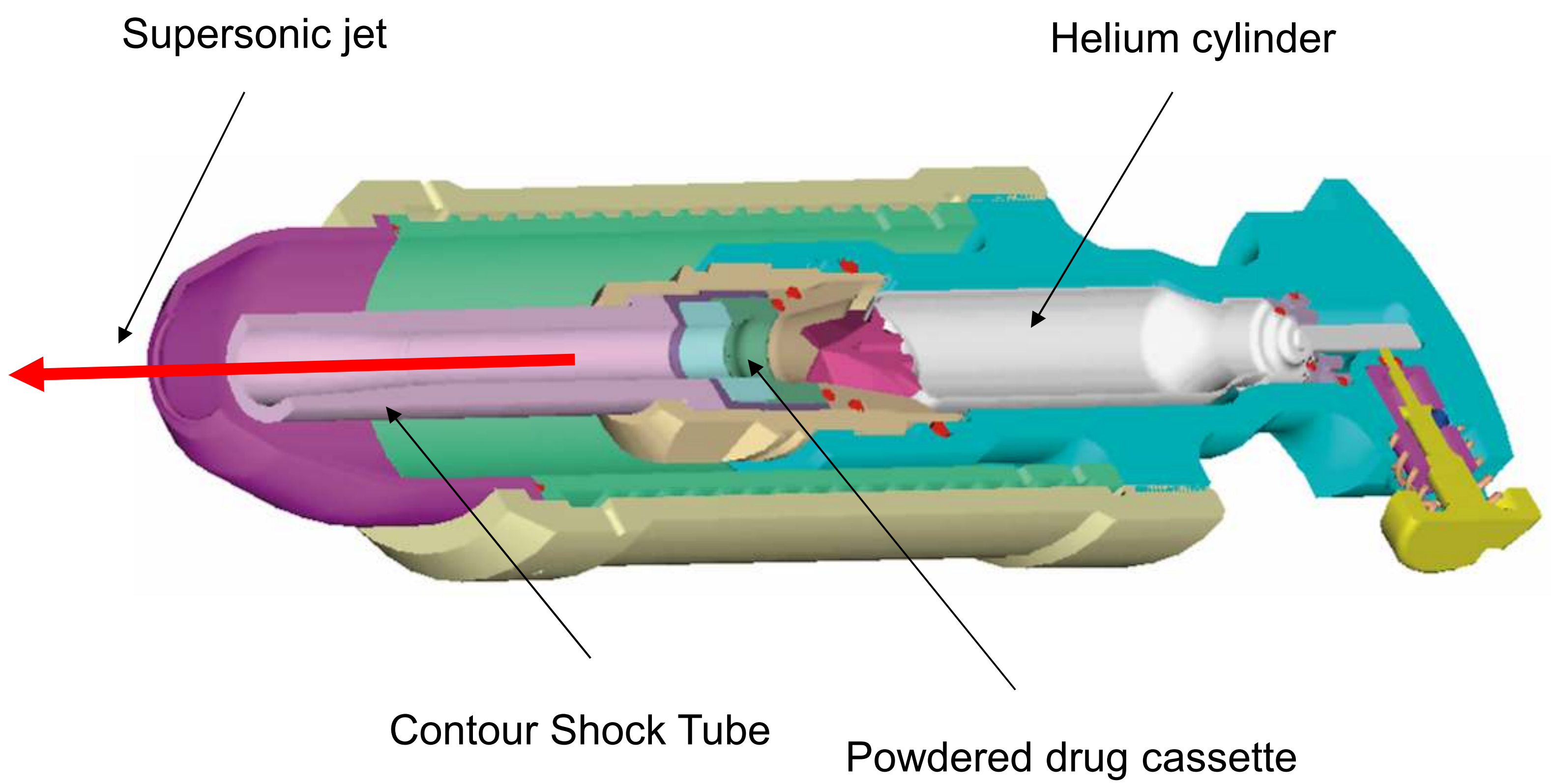
Maximum Mass Flow Rate

Since the maximum velocity at the throat is $N_{Ma} = 1$, there is a maximum flow rate:

$$\dot{m}_{max} = S^* \sqrt{\gamma \rho_0 p_0 \left[\frac{2}{\gamma + 1} \right]^{(\gamma+1)/(\gamma-1)}}$$

Increase flow by making throat larger, increasing stagnation pressure, or decrease stagnation temperature. Receiver conditions do not affect mass flow rate.

Drug Injection via Converging / Diverging Nozzle



Shock Behavior

Isentropic Paths

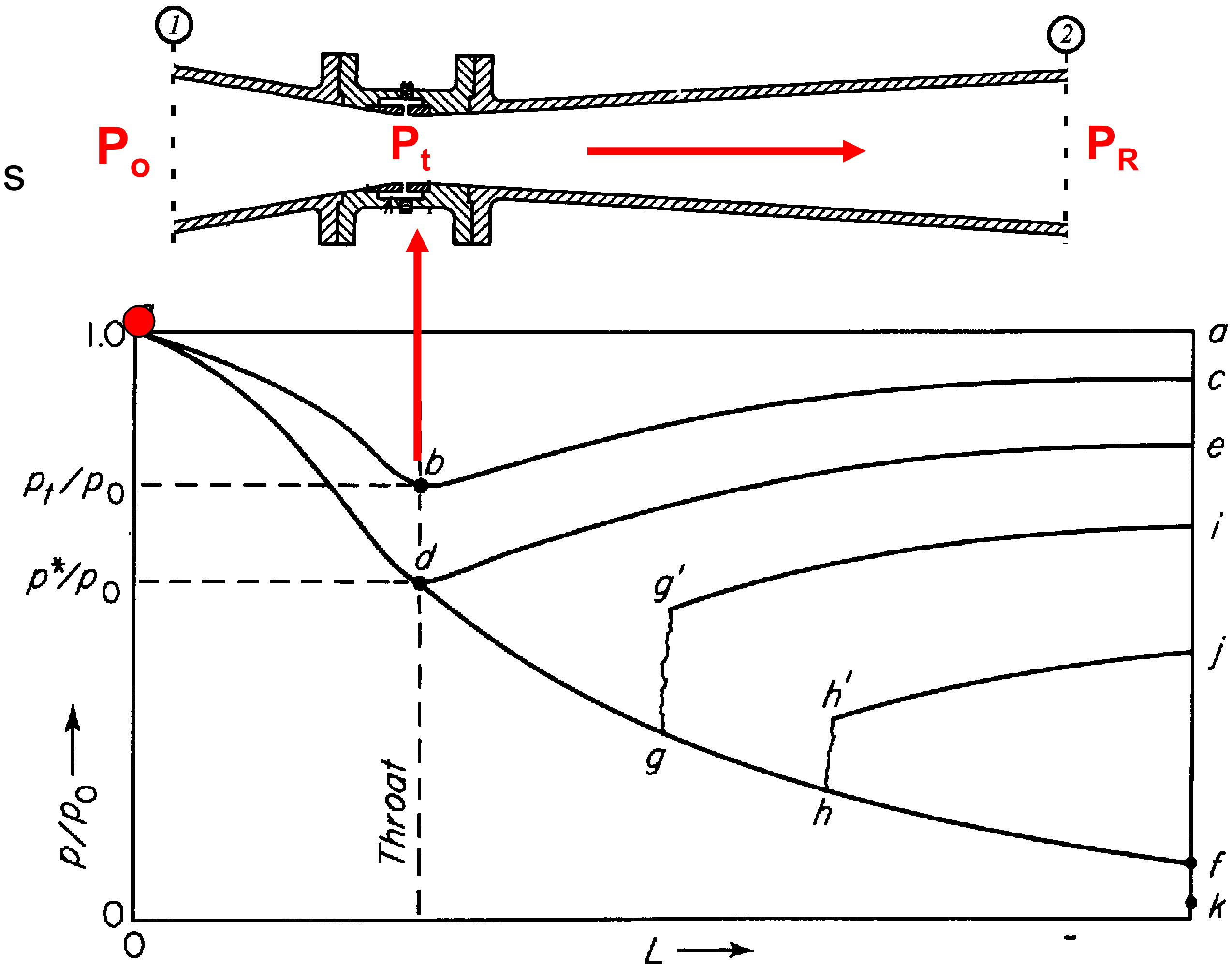
$P_R = P_c$

$P_R = P_e$

$P_R = P_f$

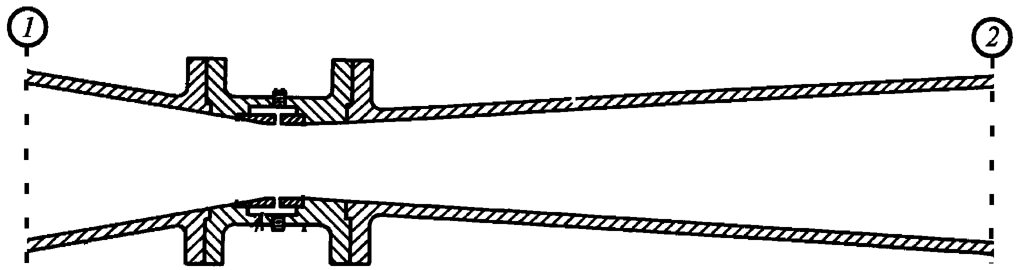
Non-Isentropic Paths

$P_e < P_R < P_f$



Sonic Flow at throat (maximum mass flowrate)

10 Minute Problem



Air flows from a large supply tank at 300 F and 20 atm (absolute) through a converging-diverging nozzle. The cross-sectional area of the throat is 1 ft2 and the velocity at the throat is sonic. A normal shock occurs at a point in the diverging section of the nozzle where the cross-sectional area is 1.18 ft2. The Mach number just after the shock is 0.70.

What would be the pressure (P1) at S = 1.18 ft^2 if no shock occurred ?

What are the new conditions (T2 and P2) after the shock ?

What is the Mach number and pressure at a point in the diverging section of the nozzle where the cross-sectional area is 1.8 ft^2 ?