system *possesses* values for its physical quantities in a way that is analogous to that in classical physics. We shall derive an inequality that is satisfied by certain correlation functions in any such theory which is also local. We shall then see how the predictions of quantum theory can violate this inequality. Since Bell's original work, many examples of 'Bell inequalities' have been discovered, and the one employed here is chosen because of its simplicity (see Redhead (1989) for more discussion of this particular example).

The considerations of EPR were concerned with two observers who make measurements along the same axis. John Bell found his famous inequalities by asking what happens if the observers measure the spin of the particles along different axes. In particular, we consider a pair of unit vectors $\mathbf{a}$ and $\mathbf{a}'$ for one observer, and another pair $\mathbf{b}$ and $\mathbf{b}'$ for the other. Now suppose a series of repeated measurements is made on a collection of systems whose quantum state is described by the vector $|\psi\rangle$ in Eq. (9.29); for example we could look at a series of decays, each of which produces a pair of particles with zero total spin angular momentum. The central realist assumption we are testing is that each particle has a definite value at all times for any direction of spin. We let a_n denote $2/\hbar$ times the value of $\mathbf{a}\cdot\mathbf{S}$ possessed by particle 1 in the n'th element of the collection. Thus $a_n = \pm 1$ if $\mathbf{a}\cdot\mathbf{S} = \pm\frac{1}{2}\hbar$.

The key ingredient in the derivation of the Bell inequalities is the correlation between measurements made by the two observers along these different directions. For directions $\mathbf{a}$ and $\mathbf{b}$ this is defined by

$$C(\mathbf{a},\mathbf{b}) := \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} a_n b_n, \tag{9.31}$$

and similarly for the other directions. Note that if the results are always totally correlated then $C(\mathbf{a},\mathbf{b}) = +1$, whereas if they are totally anti-correlated we get $C(\mathbf{a},\mathbf{b}) = -1$.

Now look at the quantity

$$g_n := a_n b_n + a_n b'_n + a'_n b_n - a'_n b'_n. \tag{9.32}$$

For any member n of the collection, each term in this sum will take on the value $+1$ or -1. Furthermore, the fourth term on the right hand side is equal to the product of the first three (because $(a_n)^2 = 1 = (b_n)^2$). Then thinking about the various possibilities shows that g_n can take on only the values ± 2. Therefore, the right hand side of the expression

$$\left|\frac{1}{N}\sum_{n=1}^{N} g_n\right| = \left|\frac{1}{N}\sum_{n=1}^{N} a_n b_n + \frac{1}{N}\sum_{n=1}^{N} a_n b'_n + \frac{1}{N}\sum_{n=1}^{N} a'_n b_n - \frac{1}{N}\sum_{n=1}^{N} a'_n b'_n\right|, \tag{9.33}$$

representing the average value of g_n, must be less than or equal to 2. Thus, in the limit as $N\to\infty$, we get

$$|C(\mathbf{a},\mathbf{b}) + C(\mathbf{a},\mathbf{b}') + C(\mathbf{a}',\mathbf{b}) - C(\mathbf{a}',\mathbf{b}')| \le 2, \tag{9.34}$$

which is one of the famous Bell inequalities.

It is important to emphasise that the only assumptions that have gone into proving Eq. (9.34) are:

1. For each particle it is meaningful to talk about the actual values of the projection of the spin along any direction.
2. There is locality in the sense that the value of any physical quantity is not changed by altering the position of a remote piece of measuring equipment. This means that both occurrences of a_n in Eq. (9.33) have the same value, *i.e.*, they do not depend on the direction ($\mathbf{b}$ or $\mathbf{b}'$) along which the other observer chooses to measure the spin of particle 2. In particular, we are ruling out the type of context-dependent values that arose in our discussion of the Kochen–Specker theorem.

We shall now show that the predictions of quantum theory violate this inequality over a range of directions for the spin measurements. The quantum-mechanical prediction for the correlation between the spin measurements along axes $\mathbf{a}$ and $\mathbf{b}$ is

$$C(\mathbf{a},\mathbf{b}) := \left(\frac{2}{\hbar}\right)^2 \langle\psi|\,\mathbf{a}\cdot\widehat{\mathbf{S}}_{(1)}\otimes\mathbf{b}\cdot\widehat{\mathbf{S}}_{(2)}\,|\psi\rangle \tag{9.35}$$

where $\widehat{\mathbf{S}}_{(1)}$ and $\widehat{\mathbf{S}}_{(2)}$ are the spin operators for particles 1 and 2 respectively, and the tensor product is defined as in Eq. (8.30). Since the total angular momentum of the vector $|\psi\rangle$ in Eq. (9.29) is zero, it is invariant under the unitary operators which generate rotations of coordinate systems (*cf.* Sections 7.1–7.2). This means that $C(\mathbf{a},\mathbf{b})$ is a function of $\cos\theta_{ab} := \mathbf{a}\cdot\mathbf{b}$ only, and hence there is no loss of generality in assuming that $\mathbf{a}$ points along the z-axis and that $\mathbf{b}$ lies in the x–z plane. Then Eq. (9.35) becomes

$$C(\mathbf{a},\mathbf{b}) = \langle\psi|\,\sigma_{1z}\otimes(\sigma_{2z}\cos\theta_{ab} + \sigma_{2x}\sin\theta_{ab})\,|\psi\rangle \tag{9.36}$$

where, for example, σ_{1z} is the z-direction Pauli spin matrix for the first particle. It is now a straightforward calculation [Exercise] to show that

$$C(\mathbf{a},\mathbf{b}) = -\cos\theta_{ab}. \tag{9.37}$$

Now let us restrict our attention to the special case in which (i) the four vectors $\mathbf{a},\mathbf{a}',\mathbf{b},\mathbf{b}'$ are coplanar; (ii) $\mathbf{a}$ and $\mathbf{b}$ are parallel; and (iii) $\theta_{ab'} = \theta_{a'b} = \phi$ say. Then the Bell inequality will be satisfied provided

$$|1 + 2\cos\phi - \cos 2\phi| \le 2. \tag{9.38}$$

However, from the form of this function of ϕ sketched in Figure 9.1 we see at once that the inequalities are *violated* for all values of ϕ between 0^0 and 90^0.