Proposition: If f is a continuous function such that $f : [0,1] \to [0,1]$, $\exists x \in [0,1] : f(x) = x$.

Proof: Let $g(x) = f(x) - x$. Since g is made up of continuous functions, we have that g is also continuous. Since f is defined only on the interval $[0,1]$, whereas x, as a function, has no restrictions, g is restricted to taking values on this interval.

$g(0) = f(0) - 0 = f(0) \in [0,1]$, as defined above.

$g(1) = f(1) - 1 \in [-1,0]$, since the highest value that f can take is 1, the highest value that it's difference will take is 0. Likewise, since f's lowest value is 0, it's lowest value after subtracting 1 is -1.

Since g is continuous and goes from above/on the x-axis to on/below the x-axis, there must be $x \in [0,1]$ such that $g(x) = 0$, by the Intermediate Value Theorem. However, this means that $f(x) - x = 0 \implies f(x) = x$ for some $x \in [0,1]$, as required.

QED