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E. T. Newman and A. I. Janis

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Note on the Kerr Spinning-Particle Metric*

E. T. NEWMAN AND A. I. JANIS
Physics Department, University of Pittsburgh, Pittsburgh, Pennsylvania
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It is shown that by means of a complex coordinate transformation performed on the monopole or Schwarzschild metric one obtains a new metric (first discovered by Kerr). It has been suggested that this metric be interpreted as that arising from a spinning particle. We wish to suggest a more complicated interpretation, namely that the metric has certain characteristics that correspond to a ring of mass that is rotating about its axis of symmetry. The argument for this interpretation comes from three separate places: (1) the metric appears to have the appropriate multipole structure when analyzed in the manner discussed in the previous paper, (2) in a covariantly defined flat space associated with the metric, the Riemann tensor has a circular singularity, (3) there exists a closely analogous solution of Maxwell's equations that has characteristics of a field due to a rotating ring of charge.

INTRODUCTION

RECENTLY, R. Kerr[1] has derived a new solution of the empty-space Einstein field equations which in some sense represents a spinning object with mass; its linearized version is a mass monopole plus the Lens–Thirring spinning-particle metric. The present note has two purposes. In the first section we give a curious "derivation" of the Kerr metric by performing a complex coordinate transformation on the Schwarzschild metric. In the second section we attempt to argue that the Kerr metric has certain characteristics that suggest a metric arising from a ring of mass rotating about its axis of symmetry. There are three points to the argument: (a) In a covariantly defined flat space, the Riemann tensor considered as a field defined on the flat space is singular on a ring, (b) there is a very close analogy between the Kerr metric and a solution of Maxwell's equations having characteristics of a rotating ring of charge, and (c) using the definitions of the gravitational multipoles given in the previous paper[2] it is seen that the Kerr metric is compatible with the structure of a rotating ring of mass.

"DERIVATION" OF KERR METRIC

The Schwarzschild metric, written in standard coordinates, is

$$ds^2 = (1 - r_0/r)\, dx^{0^2} - (1 - r_0/r)^{-1}\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \qquad r_0 = 2km/c^2. \tag{1}$$

It can be transformed by the coordinate transformation

$$u = t - r - r_0 \ln(r - r_0), \qquad r' = r,$$
$$\theta' = \theta, \qquad \phi' = \phi,$$

into the form (dropping the primes)

$$ds^2 = (1 - r_0/r)\, du^2 + 2\, du\, dr - r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{2}$$

(The surface u = constant is a spherically symmetric null surface.)

The contravariant components of the metric [Eq. (2)], namely

$$g^{00} = 0, \quad g^{11} = -(1 - r_0/r), \quad g^{12} = 1, \quad g^{22} = -1/r^2, \quad g^{33} = -1/r^2 \sin^2\theta,$$

can be written in the alternate form

$$g^{\mu\nu} = l^\mu n^\nu + l^\nu n^\mu - m^\mu \bar{m}^\nu - m^\nu \bar{m}^\mu, \tag{3}$$

where

$$l^\mu = \delta_1^\mu, \quad n^\mu = \delta_0^\mu - \tfrac{1}{2}(1 - r_0/r)\delta_1^\mu$$
$$m^\mu = \frac{1}{\sqrt{2}\, r}\left(\delta_2^\mu + \frac{i}{\sin\theta}\delta_3^\mu\right), \tag{4}$$
$$\bar{m}^\mu = \frac{1}{\sqrt{2}\, r}\left(\delta_2^\mu - \frac{i}{\sin\theta}\delta_3^\mu\right).$$

This complex null tetrad system forms the starting point of the "derivation" of the Kerr metric. "Derivation" is put in quotation marks because there is no simple, clear reason for the series of operations performed on the tetrad to yield a new (different from Schwarzschild) solution of the Einstein equations, or even to yield a solution of the empty-space equations at all. Nevertheless, we do obtain a new solution.

[Kerr has recently shown (in a private communication), from the Einstein field equations, that this

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[1] R. P. Kerr, Phys. Rev. Letters **11**, 237 (1963).
[2] A. I. Janis and E. T. Newman, J. Math. Phys. **6**, 902 (1965).

type of operation works for the class of solutions, $g_{\mu\nu} = \eta_{\mu\nu} + \lambda^2 l_\mu l_\nu$. This class contains the Schwarzschild metric as a special case.]

The coordinate r is allowed to take complex values and the tetrad is rewritten in the form

$$l^\mu = \delta_1^\mu, \quad n^\mu = \delta_0^\mu - \frac{1}{2}\left[1 - \frac{r_0}{2}\left(\frac{1}{r} + \frac{1}{\bar{r}}\right)\right]\delta_1^\mu,$$
$$m^\mu = \frac{1}{\sqrt{2}\,\bar{r}}\left(\delta_2^\mu + \frac{i}{\sin\theta}\,\delta_3^\mu\right), \tag{5}$$
$$\bar{m}^\mu = \frac{1}{\sqrt{2}\,r}\left(\delta_2^\mu - \frac{i}{\sin\theta}\,\delta_3^\mu\right),$$

$\bar{r}$ being the complex conjugate of r. (Note that part of the algorithm is to keep l^μ and n^μ real and m^μ and $\bar{m}^\mu$ the complex conjugates of each other.) We now formally perform the complex coordinate transformation

$$r' = r + ia\cos\theta, \qquad \theta' = \theta,$$
$$u' = u - ia\cos\theta, \qquad \phi' = \phi, \tag{6}$$

on the vectors l^μ, n^μ, and m^μ. ($\bar{m}'^\mu$ is defined as the complex conjugate of m'^μ.)

If one now allows r' and u' to be real, we obtain the following tetrad:

$$l'^\mu = \delta_1^\mu, \quad n'^\mu = \delta_0^\mu - \tfrac{1}{2}\{1 - r_0[r'/(r'^2 + a^2\cos^2\theta)]\}\delta_1^\mu,$$
$$m'^\mu = [\sqrt{2}\,(r' + ia\cos\theta)]^{-1} \times [ia\sin\theta(\delta_0^\mu - \delta_1^\mu) + \delta_2^\mu + (i/\sin\theta)\delta_3^\mu]. \tag{7}$$

The metric $g'^{\mu\nu} = l'^\mu n'^\nu + l'^\nu n'^\mu - m'^\mu\bar{m}'^\nu - m'^\nu\bar{m}'^\mu$ can now be shown by a coordinate transformation to be equivalent to that of Kerr.

INTERPRETATION

The contravariant form of the Kerr metric can be written as (dropping the primes)

$$g^{\mu\nu} = g^{0\mu\nu} + \lambda^2 l^\mu l^\nu,$$
$$\lambda^2 = \frac{r_0 r}{r^2 + a^2\cos^2\theta}, \quad l^\mu = \delta_1^\mu, \tag{8}$$

the $g^{0\mu\nu}$ being easily computed from Eq. (7). The form is dictated by requiring $g^{0\mu\nu}$ to be independent of r_0 (or the mass m). By calculating the Riemann tensor it is seen that if r_0 goes to zero, then the space is flat, which proves that $g^{0\mu\nu}$ can be looked on as a flat-space metric tensor which is covariantly defined by the Kerr metric. Another result from the study of the Riemann tensor is that the space is algebraically special, Petrov type ID, l_μ being one of the double principal null vectors. The vector l_μ is not, as it is in the Schwarzschild case, surface forming or hypersurface orthogonal, the constant a giving a measure of the curl of l_μ. There are two real invariants (or one complex one) which can be computed from the Riemann tensor, namely

$$\Psi_2 \equiv R_{\alpha\beta\gamma\delta}l^\alpha m^\beta n^\gamma \bar{m}^\delta = -r_0/2(r - ia\cos\theta)^3. \tag{9}$$

It should be emphasized that one can not treat the coordinates r, θ, and ϕ as usual polar coordinates, for even in the flat-space limit ($r_0 = 0$), the metric $g^{\mu\nu}$ is not the polar coordinate version of $\eta_{\mu\nu}$, the Minkowski metric. However the following coordinate transformation does lead to polar coordinates, $\tilde{r}$, $\tilde{\theta}$, $\tilde{\phi}$, and $\tilde{u} \equiv t - \tilde{r}$:

$$\tilde{r}^2 = r^2 + a^2\sin^2\theta, \quad \tan\tilde{\phi} = \frac{\tan\phi - a/r}{1 + (a/r)\tan\phi}$$
$$\cos\tilde{\theta} = r\cos\theta/(r^2 + a^2\sin^2\theta)^{\frac{1}{2}}, \tag{10}$$
$$\tilde{u} = u - (r^2 + a^2\sin^2\theta)^{\frac{1}{2}} + r.$$

We have the situation that the Kerr metric has associated with it a flat-space metric $g^{0\mu\nu}$ which allows us to define polar coordinates in the original nonflat space. We can now ask where, as a function of the polar coordinates plotted in the associated flat space, is the Riemann tensor, or its invariants, singular. Clearly Ψ_2 is singular at $r = 0$ and $\theta = \frac{1}{2}\pi$, or in polar coordinates [from Eq. (10)] it is singular on the circle $\tilde{r} = a$, $\tilde{\theta} = \frac{1}{2}\pi$. It is reasonable then to associate with the Kerr metric this ring singularity.

The second point of our interpretation arises from noting the striking analogy between the Kerr metric and a solution of the Maxwell equations. First we will show the analogy between the Schwarzschild metric and the Coulomb field. The single invariant of the Schwarzschild Riemann tensor is $\Psi_2 = -r_0/2r^3$; the analogous invariant[3] of the Coulomb field is

$$\Phi_1 \equiv \tfrac{1}{2}F_{\mu\nu}(l^\mu n^\nu + \bar{m}^\mu m^\nu) = -e/2r^2.$$

If in these two invariants we substitute $r = r' - ia\cos\theta$ [obtained from Eq. (6)], we get the invariant Eq. (9) for the Kerr metric and (dropping the prime again) for the Maxwell field we get the invariant

$$\Phi_1 = -e/2(r - ia\cos\theta)^2.$$

It can be shown that this is a solution of Maxwell's equations expressed in terms of the original coordinate system of the Kerr solution [Eq. (8) with $r_0 = 0$]; i.e., r and θ are not polar coordinates.

[3] For a discussion of the invariants of the Riemann tensor and the Maxwell field tensor and the analogy between them see Ref. 2.

Using the coordinate transformation to polar coordinates, Eq. (10), we can see that this solution is singular only on the circle $\tilde{r} = a$, and $\tilde{\theta} = \frac{1}{2}\pi$. The task of analyzing the multipole structure of this solution was rather laborious and only the first three terms were calculated with the following results: (a) the monopole moment is e; (b) there is no electric dipole and the magnetic dipole moment is proportional to ea; (c) there is no magnetic quadrupole moment and the electric quadrupole moment is proportional to ea^2; (d) there appears to be an alternation back and forth between the electric and magnetic type poles.

This structure plus its singularity leads us to conclude that the field is due to a ring of charge rotating about its axis of symmetry with angular velocity proportional to a.

The analogy between the Kerr solution and this solution of Maxwell's equations suggests that the Kerr metric represents a ring of mass rotating about its symmetry axis. This is substantiated by analyzing the multipole structure of the metric in terms of the definitions given in the previous paper.[2] The method consists of finding null surfaces in the Kerr space and introducing them as coordinate surfaces with an associated null tetrad system. A lengthy but not difficult calculation leads to results similar to that found in the Maxwell case; there exists (a) a monopole moment equal to m; (b) no mass dipole but a spin-pole proportional to ma; (c) no spin quadrupole but a mass quadrupole proportional to ma^2.

From these three points we believe that our interpretation of the Kerr metric is reasonable.[4]

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[4] We wish to thank the referee and R. P. Kerr for pointing out that, although our analysis is correct, the final interpretation of the solutions (both Maxwell and Einstein fields) is probably incorrect due to the unusual (multivalued) behavior of the solutions when a closed loop that threads the singular ring is followed. In order to avoid this multivalued behavior of the solutions, it would be necessary to have a surface distribution of matter (charge in the Maxwell case), the surface being bounded by the singular ring, in such a way that the fields are discontinuous across the surface.