Journal of Mathematical Physics

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Citation: J. Math. Phys. **6**, 918 (1965); doi: 10.1063/1.1704351
View online: http://dx.doi.org/10.1063/1.1704351
View Table of Contents: http://jmp.aip.org/resource/1/JMAPAQ/v6/i6
Published by the American Institute of Physics.

Metric of a Rotating, Charged Mass*

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(19 June 1964)

A new solution of the Einstein–Maxwell equations is presented. This solution has certain characteristics that correspond to a rotating ring of mass and charge.

THE purpose of the present note is to present a new solution of the Einstein–Maxwell equations which in some sense represents a rotating mass and charge. This solution bears the same relation to the charged Schwarzschild metric[1] (Reissner–Nordström) as the Kerr spinning particle metric bears to the Schwarzschild. In fact one can "derive" it by means of a similar trick (complex coordinate transformation) as was used to "derive" the Kerr metric.[2]

The Reissner–Nordström metric in null coordinates[2] has the form

$$ds^2 = (1 - 2m/r + e^2/r^2)\, du^2 + 2\, du\, dr - r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \qquad (1)$$

where m and e are the mass and charge respectively and u labels the null surfaces. The contravariant form of the metric can be written

$$g^{\mu\nu} = l^\mu n^\nu + l^\nu n^\mu - m^\mu \bar{m}^\nu - m^\nu \bar{m}^\mu, \qquad (2)$$

where

$$l^\mu = \delta_1^\mu, \quad m^\mu = (1/\sqrt{2}\, r)[\delta_2^\mu + (i/\sin\theta)\delta_3^\mu] \qquad (3)$$
$$n^\mu = \delta_0^\mu - \left(\frac{1}{2} - \frac{m}{r} + \frac{e^2}{2r^2}\right)\delta_1^\mu,$$

and where $\bar{m}^\mu$ is the complex conjugate of m^μ.

A new metric can now be obtained by the following formal process. The radial coordinate r is allowed to take complex values and the tetrad is rewritten in the form

$$l^\mu = \delta_1^\mu, \quad m^\mu = (1/\sqrt{2}\, \bar{r})[\delta_2^\mu + (i/\sin\theta)\delta_3^\mu] \qquad (4)$$
$$n^\mu = \delta_0^\mu - \frac{1}{2}\left(1 - m\left[\frac{1}{r} + \frac{1}{\bar{r}}\right] + \frac{e^2}{r\bar{r}}\right)\delta_1^\mu,$$

$\bar{r}$ being the complex conjugate of r. [It should be noted that if the term $e^2/2r^2$ in n^μ was replaced by $\frac{1}{4}e^2(r^{-2} + \bar{r}^{-2})$ instead of $e^2/2r\bar{r}$, we would not obtain a solution of the Einstein–Maxwell equations.] If we now perform the same complex coordinate transformation as was used in Ref. (2) ($r' = r + ia\cos\theta$, $u' = u - ia\cos\theta$) we obtain the following tetrad,

$$l^\mu = \delta_1^\mu, \quad m^\mu = [\sqrt{2}\,(r' + ia\cos\theta)]^{-1} \times [ia\sin\theta(\delta_0^\mu - \delta_1^\mu) + \delta_2^\mu + (i/\sin\theta)\delta_3^\mu],$$
$$n^\mu = \delta_0^\mu - [\tfrac{1}{2} - (mr' - \tfrac{1}{2}e^2)(r'^2 + a^2\cos^2\theta)^{-1}]\delta_1^\mu \qquad (5)$$

and associated metric tensor $g'^{\mu\nu} = l'^\mu n'^\nu + l'^\nu n'^\mu - m'^\mu \bar{m}'^\nu - m'^\nu \bar{m}'^\mu$.

If we take the following Maxwell field (stated in terms of the tetrad components[3] of the field tensor $F_{\mu\nu}$ rather than in terms of the tensor itself)

$$\phi_0 \equiv F_{\mu\nu} l^\mu m^\nu = 0$$
$$\phi_1 \equiv \tfrac{1}{2}F_{\mu\nu}(l^\mu n^\nu + \bar{m}^\mu m^\nu) = e/\sqrt{2}\,(r - ia\cos\theta)^2 \qquad (6)$$
$$\phi_2 \equiv F_{\mu\nu}\bar{m}^\mu n^\nu = iea\sin\theta/(r - ia\cos\theta)^3,$$

it can be shown by direct calculation that this field with the metric associated with Eq. (5) constitutes a solution of the Einstein–Maxwell equations. [We wish to point out that there was no simple algorithm which led to Eq. (6). It had to be obtained by integration.]

By arguments similar to those used in (2) we conclude that this solution represents the gravitational and electromagnetic fields of a ring of mass and charge rotating about its axis of symmetry.[4]

The Weyl tensor of this space is type II degenerate, the double null vector being l^μ. l^μ is also a principle null vector of the Maxwell tensor. l^μ is shear free but not hypersurface orthogonal, a measuring its curl.

In conclusion, we give the contra- and covariant

* Supported in part by Aerospace Research Laboratories, Office of Aerospace Research, U. S. Air Force.
[1] R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford University Press, London, 1934).
[2] E. T. Newman and A. I. Janis, J. Math. Phys. **6**, 915 (1965).
[3] A. I. Janis and E. T. Newman, J. Math. Phys. **6**, 902 (1965).
[4] We wish to thank the referee and R. Kerr for pointing out a difficulty in this interpretation. The remarks in Footnote 4, Ref. 2 apply here as well.

forms of the metric, where $x \equiv (r^2 + a^2 \cos^2\theta)^{-1}$,

$$g^{\mu\nu} = \left\| \begin{matrix} x(-a^2 \sin^2 \theta) & x(r^2 + a^2) & 0 & -xa \\ \cdot & x[2mr - (r^2 + a^2) - e^2] & 0 & xa \\ \cdot & \cdot & -x & 0 \\ \cdot & \cdot & \cdot & x(-\sin^{-2} \theta) \end{matrix} \right\|$$

and

$$g_{\mu\nu} = \left\| \begin{matrix} 1 + x(e^2 - 2mr) & 1 & 0 & x(a \sin^2 \theta)(2mr - e^2) \\ \cdot & 0 & 0 & -a \sin^2 \theta \\ \cdot & \cdot & -x^{-1} & 0 \\ \cdot & \cdot & \cdot & -\sin^2 \theta(r^2 + a^2 + ag_{03}) \end{matrix} \right\|.$$

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 6, NUMBER 6 JUNE 1965

Unified Dirac–Von Neumann Formulation of Quantum Mechanics. I. Mathematical Theory*

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(Received 25 September 1964)

In this paper the results from various areas of mathematical research which are necessary for a consistent unification of the Dirac and von Neumann formulations of quantum mechanics are collected and presented as a single synthesis. For this purpose, direct integral decompositions of Hilbert space must be introduced into Dirac's formulation of spectral theory and representation theory; true unit vectors in the direct integral decomposition spaces replace unnormalizable vectors of infinite length. It then becomes clear that families of modified Dirac projection operators are simply related to the Radon–Nikodym derivative of von Neumann spectral measures. In terms of these mathematical preliminaries a second paper will present the more physical aspects of the resulting unified formulation of quantum mechanics.

THE definitive and beautiful formulation given to quantum mechanics by Dirac[1] has the single disadvantage of requiring the introduction of unnormalizable vectors of infinite length into Hilbert space to represent eigenstates of observables having continuous spectra. This fact not only renders the theory mathematically nonrigorous, but even leads to practical difficulties in physical interpretation whenever powers and products of the functions representing such unnormalizable vectors (δ-functions) appear. The difficulties have been completely solved mathematically by the theories of von Neumann[2] and Schwartz,[3] but the methods used differ substantially from Dirac's approach and have not so far proved practical for physicists. Our purpose here is to present a consistent formulation of quantum mechanics which, while preserving the basic physically useful approach of Dirac, will do away with the need for mathematically objectionable unnormalizable vectors and will allow us at will to pass easily and rigorously from a Dirac-type formulation to the von Neumann formulation.

To do this we first need to present a spectral theory in terms of direct integral decompositions of Hilbert space and then, applying this theory, we can develop a rigorous but practical representation theory. This first paper, then, will be mainly a review and

* Work supported by a National Science Foundation fellowship.

[1] P. A. M. Dirac, *The Principles of Quantum Mechanics* (Clarendon Press, Oxford, England, 1958), 4th ed.

[2] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1955).

[3] L. Schwartz, *Théorie des Distributions* (Hermann & Cie., Paris, 1950–1951).