Existance of Solutions for First Order Differential Equations
"*A Calculus 4 Perspective*"
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Typically the topic of existance is not brought up in mathamatics until at least the 400 level. When I began to ponder existance, I scoured the web feverishly only to find a multitude of Advanced Calculus Proofs. After mulling it over, I decided to take on existance over my spring break. So, my target audience is obviously not someone in graduate school. If you have found this on the web and you are indeed a student in an "Introduction to Differential Equations" class, hopefully this will save you a boat load of time, and preserve your sanity. So, here we go!

Consider the Intial Value Problem:

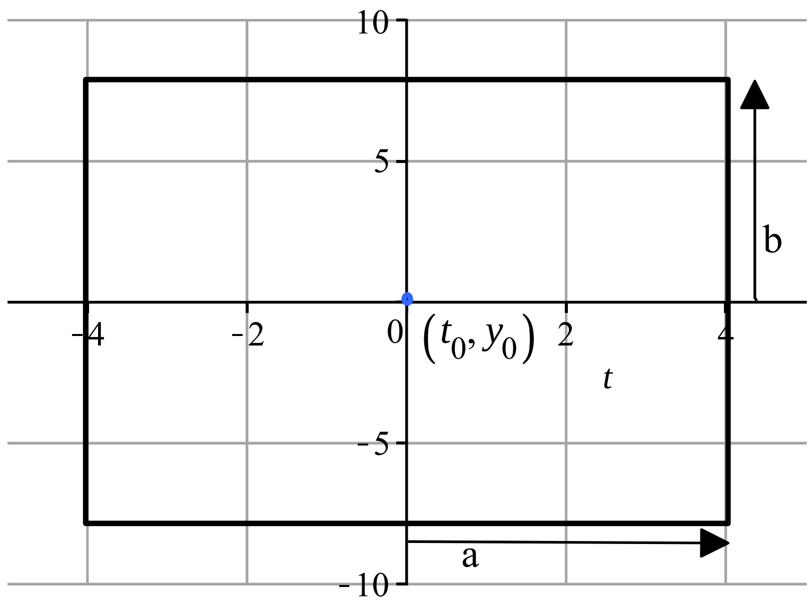
$$\frac{\mathbf{dy}}{\mathbf{dt}} = \boldsymbol{f}(\mathbf{t}, \mathbf{y}(\mathbf{t})) \qquad \mathbf{y}(\mathbf{x_0}) = \mathbf{y_0}$$

<u>Theorem:</u>
If f and $\frac{\partial f}{\partial y}$ are both continuous in some rectangle $\boldsymbol{R}$: $|t - t_0| \le a$, $|y - y_0| \le a$, then there exist a unique solution to the intial value problem for every interior point (t, y).

First things first, I will only be considering existance right now, we will worry about uniqueness later.

Proof.

To start this proof, we need to get some preliminaries out of the way. I think I should start by making sure that we are clear about the region $\boldsymbol{R}$. The rectangle defined in the theorem is centered at a point given by the initial condition, here is a graph.



So, the above is the graph of the region given by the theorem. The region above seems to be defined by: $a = 4$ **and** $b = 7.5$. The intial condition must have been $y(0) = 0$, as the rectangle is centered here.

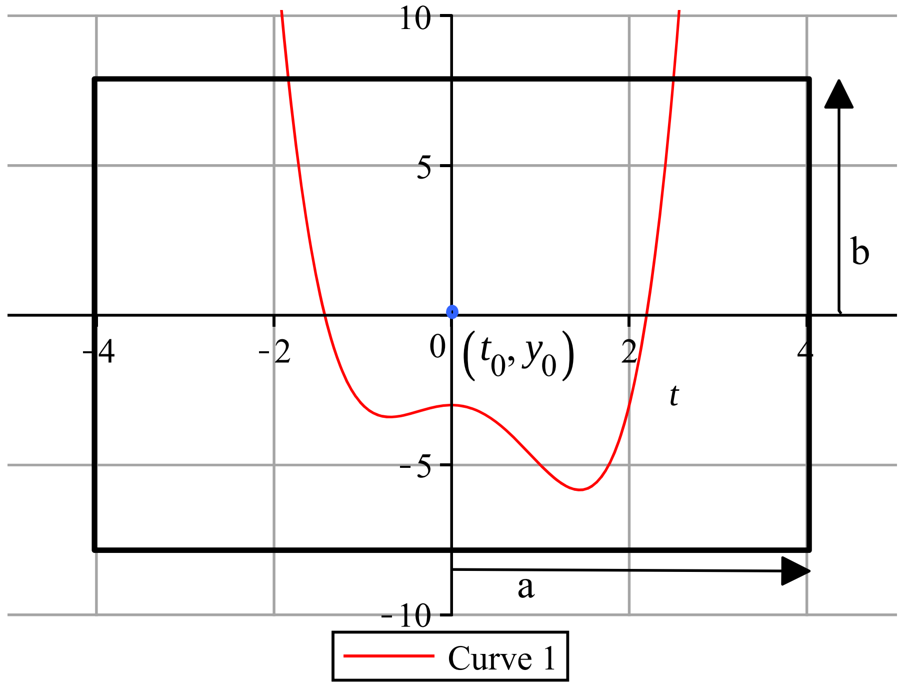
To prove the theorem above, we will now proceed with a numerical method known as Picard Iteration. The actual numerical method is not the focus of this paper. It is assumed that whomever is reading this is familiar with the method. It is not a difficult method to learn, so if you have never seen it before, you must learn the method before attempting to prove existance.

Proceeding with Picard Iteration, we have the equivalent integral equation:

$\mathbf{y_n(t) = y_0 + \int_{t_0}^{t} f(\alpha, y_{n-1}(\alpha))\, d\alpha}$, where α is just a dummy variable of integration. We wish to show that the infinite series of iterates, as $n \to \infty$ converges upon the solution to the I.V.P..

1)

The first thing we need to realize is that continuity is only assumed to exist inside the rectangle given in the theorem. It is forseeable that a problem may arise if the graph of any iterate runs outside the rectangle while being evaluated at any value t in the domain given by $|t - t_0| \le a$. To get a better sense of the problem, check out this graph:



This is the same graph given before, this time with some arbitrary iterate graphed on it. As you can see, there is indeed a problem here! We only know f to be continuous when evaluated at points (t,y) within the rectangle. The domain given by $|t| \leq 4$ is allowing y values to escape the confines of the rectangle. This is bad news, since we cannot even claim $f(t, y)$ to be existant when evaluated at any point outside of the region given in the theorem. Thankfully there is a clever way to clear up this mess. It is as follows:

<u>Assume</u> y_{n-1}, the nth iterate is located within the rectangle for the entirety of the domain given by the theorem. If this is the case, we know that $f(t, y_{n-1})$ is certainly bounded by some constant, we can call this constant τ. We can now go one step further and recognize that an absolute bound must also exist such that:

$|f(t, y_{n-1})| \leq \zeta$ where ζ is simply some constant. It is possible that τ and ζ are the same constant, but generally they are different. Now, we must have by definition:

$$f(t, y_{n-1}) = \frac{d}{dt}(y_n) \quad \Rightarrow \quad |f(t, y_{n-1})| = |\frac{d}{dt}(y_n)|$$

It follows that:

$$|\frac{d}{dt}(y_n)| \leq \zeta \quad \Rightarrow |\frac{\Delta y}{\Delta t}| = |\frac{y_n - y_0}{t - t_0}| \leq \zeta$$

We Arrive at the following:

$$\left|\mathbf{y_n - y_0}\right| \leq \zeta\left|\mathbf{t - t_0}\right|$$

It is easy to see from the expression above that we need to restrict the domain to:

$$\left|\mathbf{t - t_0}\right| \leq \frac{\boldsymbol{b}}{\zeta} \Rightarrow \left|\mathbf{y_n - y_0}\right| \leq \boldsymbol{b}$$

At this point, it is very important to recognize that we solved for the domain restriction under the **ASSUMPTION** that the iterate we started with was wholly contained within the confines of the rectangle for every value t in it's domain. This is not as big of a problem as it may seem. Assuming we take our first iterate, y_0 to be identically the intial value - the inequality obviously holds. There is now proof that - so long as we restrict the domain to $\left|\mathbf{t - t_0}\right| \leq \frac{\boldsymbol{b}}{\zeta}$, every iterate is guarenteed to remain within the confines of the rectangle, thus, the continutity of f **and** $\frac{\partial f}{\partial y}$ are also guarenteed!

From here on out, **let** $\psi = \frac{\boldsymbol{b}}{\zeta}$ **and** $\boldsymbol{R'}\!:\left|t - t_0\right| \leq \psi, \left|y - y_0\right| \leq \boldsymbol{a}$

The above is simply the restricted version of our rectangle.

Now that we have the domain issue taken care of, we can proceed!

2)

The time has come to derive the realation nessecary to prove the theorem. The relation is called a Lipshitz Condition, and we now wish to show that one exist in our situation. We will derive this condition using the Mean Value Theory from Calculus. Hopefully the following will be easily recognizable, it is slightly modified for three dimensions.

So, is the mean value theorem applicable? Recall roughly, that all it requires is that the function and it's first derivative be continuous in some common interval. As we just shown, with our new domain restriction, we can personally guarentee the continuity holds for every point (t,y) in $\boldsymbol{R'}$. It follows:

$$\frac{f(\mathrm{t}, \mathrm{y_n}) - f(\mathrm{t}, \mathrm{y_{n-1}})}{\mathrm{y_n} - \mathrm{y_{n-1}}} = \left.\frac{\partial f}{\partial \mathrm{y}}\right|_{(\mathrm{t}, \mathrm{C})} \qquad \forall\ (\mathrm{t,y}) \in \boldsymbol{R'}$$

On a side note, the notation above reads, for all points (t,y) that are an element of $\boldsymbol{R'}$. The upside down A is just short hand for - "for all".

Notice in the above that I have simply held t constant, which renders the M.V.T applicable to our situation. Now, when two things are equal, there absolutes are equal, so generally we have:

$$\left|\frac{f(t, y_n) - f(t, y_{n-1})}{y_n - y_{n-1}}\right| = \left|\frac{\partial f}{\partial y}\Big|_{(t, C)}\right| \qquad \forall\ (t,y) \in \boldsymbol{R'}$$

Using the same line of reasoning as before, if $\frac{\partial f}{\partial y}$ is bounded and continuous in our region, which it is, it must be bounded by some constant. It must also have an absolute bound within the region. Let us call that absolute bound β such that:

$$\left|\frac{\partial f}{\partial y}\right| \leq \beta \qquad \forall\ (t,y) \in \boldsymbol{R'}$$

So it follows that:

$$\left| f(t, y_n) - f(t, y_{n-1})\right| \leq \beta \left| y_n - y_{n-1}\right| \qquad \forall\ (t,y) \in \boldsymbol{R'}$$

The above inequality is known as the Lipshitz Condition.

3)

Now we may finally start looking at partial sums. The nth partial sum is below.

$$y_n = y_0 + (y_1 - y_0) + (y_2 - y_1) + \dots + (y_n - y_{n-1})$$

We now wish to approximate the values of iterates by definition:

$$(y_1 - y_0) = \int_{t_0}^{t} f(\alpha, y_0)\, d\alpha$$

Taking absolutes:

$$|y_1 - y_0| = \left|\int_{t_0}^{t} f(\alpha, y_0)\, d\alpha\right|$$

$$|y_1 - y_0| \leq \zeta \left|\int_{t_0}^{t} d\alpha\right| \quad \because \zeta \text{ is the absolute bound of } f, \text{ also, it can be pulled out of the absolutes}$$

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$$|y_1 - y_0| \leq \zeta|t - t_0| \leq \zeta\psi$$

Once again, by definition:

$$(y_2 - y_1) = \int_{t_0}^{t} f(\alpha, y_1) - f(\alpha, y_0)\,d\alpha$$

$$|y_2 - y_1| = |\int_{t_0}^{t} f(\alpha, y_1) - f(\alpha, y_0)\,d\alpha|$$

$$|\int_{t_0}^{t} f(\alpha, y_1) - f(\alpha, y_0)\,d\alpha| \leq \beta \int_{t_0}^{t} |y_1 - y_0|\, d\alpha$$

Note that in the above step, the Lipshitz Condition was applied. This is a recurring action. We use
the condition to obtain a relation with $|y_1 - y_0|$ as follows:

$$|y_2 - y_1| \leq \beta \int_{t_0}^{t} |y_1 - y_0|\, d\alpha \leq \beta\,\zeta \int_{t_0}^{t} |\alpha - t_0|\, d\alpha \leq \frac{\beta\,\zeta\,|t - t_0|^2}{2}$$

$$|y_2 - y_1| \leq \frac{\beta\zeta\psi^2}{2}$$

The Lipshitz Condition is integrated above which seems off. However, if you consider its orgins in the Mean Value Theorem, one soon comes to the conclusion that the inequality simply represents
some constants in disguise. Remember that the L.C. describes a relation between points, and it is numerical in nature. When we integrate both sides, the relation holds, the inequality is still true because exactly the same operations on exactly the same interval were performed on the constants.
The Lipshitz constant β can even be seen as the same. The point is, I don't think we should still call
it a Lipshitz Condition after we integrate it, but the relation is still true. One side is still less than the
other, and that is what is important to us.

As you calculate more and more iterates, the pattern begins to emerge. By mathematical induction:

$$\left|y_n - y_{n-1}\right| \leq \frac{\beta^{n-1} \cdot \zeta \cdot \psi^n}{n!}$$

4)

Now we just need to put it all together! Once again, lets look at the nth partial sum:

$$y_n = y_0 + (y_1 - y_0) + (y_2 - y_1) + ... + (y_n - y_{n-1})$$

Taking absolutes:

$$|y_n| = |y_0 + (y_1 - y_0) + (y_2 - y_1) + ... + (y_n - y_{n-1})|$$

$$|y_n| \le |y_0| + |(y_1 - y_0)| + |(y_2 - y_1)| + ... + |(y_n - y_{n-1})| \quad \because \quad |x + y| \le |x| + |y|$$

$$|y_n| \le |y_0| + |(y_1 - y_0)| + |(y_2 - y_1)| + ... + |(y_n - y_{n-1})| \le$$

$$|y_0| + \zeta\psi + \frac{\beta \cdot \zeta \cdot \psi^2}{2} + ... + \frac{\beta^{n-1} \cdot \zeta \cdot \psi^n}{n!}$$

So we can ascertain:

$$|y_n| = \left| y_0 + \sum_{n=1}^{\infty} (y_n - y_{n-1}) \right| \le |y_0| + \sum_{n=1}^{\infty} \frac{\beta^{n-1} \cdot \zeta \cdot \psi^n}{n!} \qquad \forall (t) \in \boldsymbol{R'}$$

By the ratio test, the larger series can easily be shown absolutely convergent. Since the absolute of y_n is bounded above by a convergent series, the series must also converge by comparison. Thus, in $\boldsymbol{R'}$, the series of iterates converges upon the actual solution to the I.V.P.!

Hence, Existance is Proven!