

# 5-Dimensional Kaluza-Klein Theory with a Source

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## Abstract

A free test particle in 5-dimensional Kaluza-Klein spacetime will show its electricity in the reduced 4-dimensional spacetime when it moves along the fifth dimension. In the light of this observation, we study the coupling of a 5-dimensional dust field with the Kaluza-Klein gravity. It turns out that the dust field can curve the 5-dimensional spacetime in such a way that it provides exactly the source of the electromagnetic field in the 4-dimensional spacetime after the dimensional reduction.

Keywords: Kaluza-Klein theory; source; dimensional reduction.

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## 1 Introduction

A unified formulation of Einstein's theory of gravitation and Maxwell's theory of electromagnetism in 4-dimensional spacetime was first proposed by Kaluza and Klein using 5-dimensional geometry[1][2]. The original Kaluza-Klein (K-K) theory assumed the so-called "*Cylinder Condition*", which means that there exists a spacelike Killing vector field  $\xi^a$  on the 5-dimensional spacetime  $(\hat{M}, \hat{g}_{ab})$  [3]-[5]. In addition, Kaluza demands also that  $\xi^a$  is normalized, i.e.,

$$\phi \equiv \hat{g}_{ab} \xi^a \xi^b = 1. \quad (1)$$

Note that the abstract index notation[6] is employed throughout the paper and the signature of the 5-metric is of the convention  $(-, +, +, +, +)$ . Later research shows that the ansatz (1) may be dropped out and the  $\phi$  may play a key role in the study of cosmology[7]-[10]. Being the extra dimension, the orbits of  $\xi^a$  are geometrically circles. The physical consideration that any displacement in

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the usual physical 4-dimensional spacetime should be orthogonal to the extra dimension implies that the "physical" 4-dimensional metric should be defined as:

$$g_{ab} = \hat{g}_{ab} - \phi^{-1} \xi_a \xi_b. \quad (2)$$

For practical calculation, it is convenient to take a coordinate system  $\{z^M = (x^\mu, y) | \mu = 0, 1, 2, 3\}$  with coordinate basis  $(e_M)^a = \{(e_\mu)^a, (e_5)^a\}$  on  $\hat{M}$  adapted to  $\xi^a$ , i.e.,  $(e_5)^a = (\frac{\partial}{\partial y})^a = \xi^a$ . Then the 5-metric components  $\hat{g}_{MN}$  take the form:

$$\hat{g}_{MN} = \begin{pmatrix} g_{\mu\nu} + \phi B_\mu B_\nu & \phi B_\mu \\ \phi B_\nu & \phi \end{pmatrix}, \quad (3)$$

where  $\hat{g}_{\mu 5} \equiv \phi B_\mu$ . So, locally, the "physical" spacetime can be understood as a 4-manifold  $M$  with the coordinates  $\{x^\mu\}$  endowed with the metric  $g_{ab}$ .

The whole theory is governed by the 5-dimensional Einstein-Hilbert action:

$$S_G = -\frac{1}{2k} \int_{\hat{M}} \sqrt{-\hat{g}} \hat{R}. \quad (4)$$

Suppose the range of the fifth coordinate to be  $0 \leq y \leq L$  and the 4-dimensional gravitational constant to be  $k = \hat{k}/L$ . Let  $B_\mu = f A_\mu$ ,  $f^2 = 2k$ , then equation (4) becomes a coupling action on  $M$  as:

$$\hat{S}_G = \int_M \sqrt{-g} \sqrt{\phi} \left[ -\frac{1}{2k} R + \frac{1}{4} \phi F_{ab}(A) F^{ab}(A) \right], \quad (5)$$

where  $R$  is the curvature scalar of  $g_{ab}$  on  $M$  and  $F_{ab}(A) \equiv 2\partial_{[a} A_{b]}$ . Thus, it results in a 4-dimensional gravity  $g_{ab}$  coupled to an electromagnetic field  $A_a$  and a scalar field  $\phi$ . It is clear that, under the ansatz (1), 5-dimensional K-K theory unifies the Einstein's gravity and the source-free Maxwell's field in the standard formulism. However, we know in the actual world that any electromagnetic fields as well as curved geometries are necessarily caused by some sources. Therefore, to arrive at a realistic K-K theory, one has to find the source in  $\hat{M}$ , which is responsible for the electromagnetic field in  $M$ . While K-K theory has been investigated for so many years, to our knowledge the above question has not been discussed in the literature so far.

In section 2, the dynamics of a classical test particle in 5-dimensional K-K spacetime is considered. It is shown that the motion of the particle along the extra dimension is responsible for its electricity in the 4-dimensional spacetime. This motives us to study the coupling of a dust field with the 5-dimensional gravity in section 3. The corresponding 4-dimensional formulism of the coupling is obtained. At last, by invoking the ansatz (1), a clear physical explanation of the dust as the source for the electromagnetic field is provided.

## 2 Classical Particle in K-K Spacetime

The geodesic motion of a classical particle in K-K spacetime has been studied by many authors [11]-[15]. Here we will sketch the basic idea and try a slightly

different presentation. Let  $V^a$  be the 5-velocity of a test particle in the 5-dimensional spacetime  $(\hat{M}, \hat{g}_{ab})$ . Then we can decompose  $V^a$  into its projection  $u^a$  in the "physical" spacetime  $(M, g_{ab})$  and another vector  $u_\perp^a$  which represents its motion along the fifth dimension, i.e.,

$$u^a \equiv g_b^a V^b = V^\mu (e_\mu)^a - (B_\mu V^\mu)(e_5)^a, \quad (6)$$

$$u_\perp^a \equiv V^a - u^a = (V^5 + B_\mu V^\mu)(e_5)^a, \quad (7)$$

where  $g_b^a \equiv \delta_b^a - \phi^{-1} \xi^a \xi_b$  is the projective map which can project any 5-vector into a vector in  $M$ . Since  $g_{ab}(e_5)^b = 0$ , the 4-velocity of the particle in  $M$  can be defined as:

$$v^a \equiv \frac{V^\mu (e_\mu)^a}{\sqrt{-V^\mu V_\mu}}, \quad (8)$$

where  $V^\mu V_\mu \equiv g_{\mu\nu} V^\mu V^\nu$ . The worldline of a free particle in  $(\hat{M}, \hat{g}_{ab})$  is a timelike geodesic satisfying:

$$\frac{d^2 z^M}{d\tau^2} + \hat{\Gamma}_{NR}^M \frac{dz^N}{d\tau} \frac{dz^R}{d\tau} = 0, \quad (9)$$

where  $\hat{\Gamma}_{NR}^M$  is the Christoffel symbol of  $\hat{g}_{MN}$  and  $dz^M/d\tau = V^M$ . Eq. (9) can be rewritten as:

$$\frac{dV^\mu}{d\tau} + \hat{\Gamma}_{NR}^\mu V^N V^R = 0, \quad (10)$$

$$\frac{dV^5}{d\tau} + \hat{\Gamma}_{NR}^5 V^N V^R = 0. \quad (11)$$

It is not difficult to see that Eq.(11) implies [11]

$$\frac{dQ}{d\tau} \equiv V^M \partial_M [\phi(B_\rho V^\rho + V^5)] = 0. \quad (12)$$

From Eq.(3) one knows that the line element of  $\hat{g}_{ab}$  reads

$$d\hat{s}^2 = ds^2 + \phi(B_\mu dx^\mu + dy)^2, \quad (13)$$

where  $ds^2$  is the line element of  $g_{ab}$ . Hence from Eqs. (13) and (12) we can obtain the relation of the particle's proper time  $\tau$  in  $\hat{M}$  and proper time  $t$  in  $M$  as

$$dt = d\tau \sqrt{1 + Q^2/\phi}. \quad (14)$$

In the light of Eq.(14), one can further show that Eq.(10) leads to

$$v^\rho \nabla_\rho v^\mu - \frac{fQ}{\sqrt{1 + Q^2/\phi}} F_\nu^\mu(A) v^\nu - \frac{Q^2}{2(1 + Q^2/\phi)\phi^2} (\partial^\mu \phi + v^\mu v^\nu \partial_\nu \phi) = 0, \quad (15)$$

where  $\nabla_a$  is the covariant derivative of  $g_{ab}$  in  $M$ . Since  $Q$  is a constant of the motion, we can define a charge:

$$q \equiv \frac{fmQ}{\sqrt{1 + Q^2}}, \quad (16)$$

where  $m$  denotes the mass of the classical particle. When we drop the influence of the scalar field  $\phi$  by ansatz (1), Eq.(15) takes the form:

$$v^b \nabla_b v^a - \frac{q}{m} F_b^a(A) v^b = 0. \quad (17)$$

It coincides exactly with the equation for the motion of an electrically charged particle in the electromagnetic field [16], and hence  $q$  gets its physical meaning as the electric charge. Note that the electric charge  $q$  defined here is different from that in Refs. [11] and [12]. Recalling the definition of  $u_\perp^a$ , we conclude that the so-called electric charge in four dimensions is just a manifestation of a particle's motion along the fifth dimension in K-K spacetime.

### 3 A Source of K-K Spacetime

To look for the source of the electromagnetic field and the curved K-K spacetime, the result in last section motivates us to consider a 5-dimensional dust field with 5-velocity  $V^a$  in  $\hat{M}$ . It is determined by the action:

$$\hat{S}_D = - \int_{\hat{M}} \sqrt{-\hat{g}} \hat{\mu}, \quad (18)$$

where the proper energy density  $\hat{\mu}$  is adjusted to keep the fluid current vector  $j^a \equiv \hat{\mu} V^a$  conserved. The variation of action (18) with respect to the flow line of the fluid yields [17]

$$\hat{\mu} V^b \hat{\nabla}_b V^a = 0, \quad (19)$$

where  $\hat{\nabla}_b$  is the covariant derivative of  $\hat{g}_{ab}$  in  $\hat{M}$ . Hence the flow lines of the dust are timelike geodesics in the 5-dimensional spacetime. The energy-momentum tensor of the dust field reads

$$\hat{T}_{ab} = \hat{\mu} V_a V_b. \quad (20)$$

Suppose that the dust is reduced to some fluid in  $M$ . Let  $P$  be a "comoving observer" in  $M$ , with Eq.(8) as its 4-velocity. Then  $P$  corresponds to an observer  $\hat{P}$  in  $\hat{M}$  with 5-velocity

$$\hat{v}^a = \frac{u^a}{\sqrt{-V^\mu V_\mu}}, \quad (21)$$

where  $u^a$  is defined as Eq.(6). Thus the energy density of the dust with respect to  $\hat{P}$  will be

$$\bar{\mu} = \hat{T}_{ab} \hat{v}^a \hat{v}^b = \hat{\mu} (-V^\mu V_\mu). \quad (22)$$

Note that we have

$$-1 = V^a V_a = V^\mu V_\mu + \phi(V^\rho B_\rho + V^5)^2. \quad (23)$$

We now follow the arguments in last section on free particles in K-K spacetime to endow the electricity to the fluid in  $M$ . According to Eq.(16), one may define

$\bar{\rho} \equiv f\bar{\mu}Q/\sqrt{1+Q^2}$  to be the "electric charge density" of the fluid. However, the observer  $P$  can only observe 4 dimensions, so the energy density of the fluid, which  $P$  actually measures in  $M$ , is not  $\bar{\mu}$  but  $\mu = \bar{\mu}L$ . Correspondingly, the electric charge density in  $M$  is  $\rho = L\bar{\rho}$ . Similar to Eq.(15) we can obtain from Eq.(19) the reduced 4-dimensional field equation of the fluid in  $M$  as

$$\mu v^\rho \nabla_\rho v^\mu - \gamma F_\nu^\mu(A) J^\nu - \frac{\mu Q^2}{2(1+Q^2/\phi)\phi^2} (\partial^\mu \phi + v^\mu v^\nu \partial_\nu \phi) = 0, \quad (24)$$

where we have defined  $\gamma \equiv \sqrt{\phi(1+Q^2)/(\phi+Q^2)}$  and  $J^a = \rho v^a$  is the electric 4-current density.

We now consider the reduction of 5-dimensional Einstein's equation:

$$\hat{R}_{MN} - \frac{1}{2} \hat{g}_{MN} \hat{R} = \hat{k} \hat{\mu} V_M V_N, \quad (25)$$

which is equivalent to

$$\hat{R}_{MN} = \hat{k} \hat{\mu} (V_M V_N + \frac{1}{3} \hat{g}_{MN}). \quad (26)$$

It is not difficult to show from Eq.(3) that the components of the 5-dimensional Ricci tensor  $\hat{R}_{ab}$  can be expressed as [9]:

$$\hat{R}_{55} = \frac{1}{2} k \phi^2 F^{\sigma\rho} F_{\sigma\rho} - \frac{1}{2} \nabla^\mu \nabla_\mu \phi + \frac{1}{4\phi} (\nabla^\mu \phi) \nabla_\mu \phi, \quad (27)$$

$$\hat{R}_{\mu 5} = \frac{f}{2} (\phi \nabla^\nu F_{\mu\nu} + \frac{3}{2} F_{\mu\nu} \nabla^\nu \phi) + B_\mu [\frac{1}{2} k \phi^2 F^{\sigma\rho} F_{\sigma\rho} - \frac{1}{2} \nabla^\nu \nabla_\nu \phi + \frac{1}{4\phi} (\nabla^\nu \phi) \nabla_\nu \phi], \quad (28)$$

$$\begin{aligned} \hat{R}_{\mu\nu} &= R_{\mu\nu} - k \phi F_\mu^\sigma F_{\sigma\nu} - \frac{1}{2\phi} \nabla_\mu \nabla_\nu \phi + \frac{1}{4\phi^2} (\nabla_\mu \phi) \nabla_\nu \phi \\ &\quad + B_\mu B_\nu [\frac{1}{2} k \phi^2 F^{\sigma\rho} F_{\sigma\rho} - \frac{1}{2} \nabla^\sigma \nabla_\sigma \phi + \frac{1}{4\phi} (\nabla^\sigma \phi) \nabla_\sigma \phi] \\ &\quad + \frac{f}{2} B_\mu (\phi \nabla^\sigma F_{\nu\sigma} + \frac{3}{2} F_{\nu\sigma} \nabla^\sigma \phi) + \frac{f}{2} B_\nu (\phi \nabla^\sigma F_{\mu\sigma} + \frac{3}{2} F_{\mu\sigma} \nabla^\sigma \phi). \end{aligned} \quad (29)$$

Plunging Eq.(27) into Eq.(26) we obtain a coupling equation for the matter fields as

$$\frac{1}{2} k \phi^2 F^{ab} F_{ab} = \frac{1}{2} \nabla^a \nabla_a \phi - \frac{1}{4\phi} (\nabla^a \phi) \nabla_a \phi + k \mu \phi [1 - \frac{2\phi}{3(\phi+Q^2)}]. \quad (30)$$

Plunging Eq.(28) into Eq.(26) and using Eq.(30), we obtain an electromagnetic field equation with source as

$$\phi \nabla^b F_{ab} + \frac{3}{2} F_{ab} \nabla^b \phi = \gamma J_a. \quad (31)$$

Plugging Eq.(29) into Eq.(26) and using Eqs. (30) and (31), we obtain a 4-dimensional Einstein equation with source as

$$G_{ab} = k[\mu v_a v_b + \phi(F_a^c F_{cb} - \frac{1}{4}g_{ab}F^{cd}F_{cd}) + \frac{1}{k\sqrt{\phi}}(\nabla_a \nabla_b \sqrt{\phi} - g_{ab}\nabla^c \nabla_c \sqrt{\phi})], \quad (32)$$

where  $G_{ab}$  is the Einstein tensor of  $g_{ab}$ .

Now we turn to the variational principle. In the light of Eq.(23), action (18) can be written as:

$$\begin{aligned} \hat{S}_D &= \int_{\hat{M}} \sqrt{-\hat{g}} \frac{\hat{\mu}}{\sqrt{V^\mu V_\mu}} \\ &= - \int_{\hat{M}} \sqrt{-\hat{g}} \hat{\mu} (1 + \frac{\phi(V^\rho B_\rho + V^5)^2}{V^\mu V_\mu}) \\ &= - \int_{\hat{M}} \sqrt{-\hat{g}} \hat{\mu} (1 - \frac{QB_\lambda v^\lambda}{\sqrt{-V^\mu V_\mu}} + \frac{QV^5}{V^\mu V_\mu}), \end{aligned} \quad (33)$$

which is reduced to a 4-dimensional action for the fluid on  $M$  as:

$$\hat{S}_D = - \int_M \sqrt{-g} \sqrt{\phi} (\mu - \gamma J^\mu A_\mu - \frac{\gamma \rho V^5}{f \sqrt{1 + Q^2/\phi}}). \quad (34)$$

Thus the total action of K-K gravity coupled with a dust field reads:

$$\hat{S}_G + \hat{S}_D = \int_M \sqrt{-g} \sqrt{\phi} [-\frac{1}{2k}R + \frac{1}{4}\phi F_{\mu\nu}(A)F^{\mu\nu}(A) - \mu + \gamma J^\mu A_\mu + \frac{\gamma \rho V^5}{f \sqrt{1 + Q^2/\phi}}]. \quad (35)$$

Recall that the K-K spacetime admits a killing vector field  $\xi^a$ , which will correspond to a conservative vector:

$$P^a = \hat{T}^{ab}\xi_b = \hat{\mu}V^a V^b(e_5)_b = \frac{\gamma}{Lf}[J^\mu(e_\mu)^a + \frac{\rho V^5(e_5)^a}{\sqrt{1 + Q^2/\phi}}]. \quad (36)$$

It follows that  $\sqrt{-\hat{g}}P^a$  and hence the components  $\sqrt{-\hat{g}}\gamma\rho V^5/(f\sqrt{1 + Q^2/\phi})$  and  $\sqrt{-\hat{g}}\gamma J^\mu$  are invariant under the variation with respect to the metric components  $\hat{g}_{MN}$  (or  $g_{\mu\nu}, \phi$  and  $A_\mu$ )[17]. Therefore the last term in action (35) can be neglected if one only concerns the fields equations equivalent to the 5-dimensional Einstein equations. We thus get an action on  $M$  as

$$S = \int_M \sqrt{-g} \sqrt{\phi} [-\frac{1}{2k}R + \frac{1}{4}\phi F_{ab}(A)F^{ab}(A) - \mu + \gamma J^a A_a], \quad (37)$$

which coincides with the 4-dimensional coupled action of gravity, charged fluid, electromagnetic field and a scalar field. Its variations with respect to  $\phi$ ,  $A^\mu$ , and  $g^{\mu\nu}$  yield respectively Eqs. (30), (31) and (32) exactly. Note that the energy density  $\mu$  contains also the variational information of  $\hat{\mu}$  with respect to  $\phi$  whence the last two terms in action (35) are invariant.

## 4 Discussion

To reveal the physical meaning of the coupled dust field, we invoke Kaluza's ansatz (1). Then the field equations (24), (30), (31) and (32) become respectively

$$\mu v^b \nabla_b v^a = F_b^a J^b, \quad (38)$$

$$\frac{1}{2} F^{ab} F_{ab} = k\mu \left[ 1 - \frac{2}{3(1+Q^2)} \right], \quad (39)$$

$$\nabla^b F_{ab} = J_a, \quad (40)$$

$$G_{ab} = k(T_{ab}^{(D)} + T_{ab}^{(E)}), \quad (41)$$

where  $T_{ab}^{(D)} \equiv \mu v_a v_b$  and  $T_{ab}^{(E)} \equiv F_a^c F_{cb} - \frac{1}{4} g_{ab} F^{cd} F_{cd}$  are respectively the usual energy-momentum tensors of the dust and the electromagnetic fields. Subject to Eq.(39), Eqs. (38), (40) and (41) are respectively the standard 4-dimensional Lorentz, Maxwell and Einstein equations for the coupling of charged fluid, electromagnetic field and gravity. Therefore, the 4-dimensional manifestation of a dust field in K-K spacetime can be understood as the source of the reduced electromagnetic field in the "physical" spacetime, provided the dust has motion along the fifth dimension. Thus we have found the source for the electromagnetic field as well as the curved 5-dimensional spacetime. Note that a test particle can only show the effects of a spacetime on itself by its motion, but the particle does not affect the background spacetime. Hence the well-known fact about a free particle in a given K-K spacetime could not ensure our result, which concerns the effects of a dust field on the 5-dimensional spacetime. This is certainly a non-trivial physical result, although the motivation is obvious. It further completes the physical explanation of the electric charge as the motion of a free particle along the extra dimension in K-K theory.

The physical meaning of the dust field is obvious also from the viewpoint of the action. If we had imposed ansatz (1) into action (37) directly, it would become

$$S = \int_M \sqrt{-g} \left[ -\frac{1}{2k} R + \frac{1}{4} F_{ab}(A) F^{ab}(A) - \mu + J^a A_a \right], \quad (42)$$

which coincides with the standard 4-dimensional coupled action of gravity, electromagnetic field and charged fluid. This action would result in the field equations (41) and (40) without the restrictive equation (39).

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## References

- [1] T. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Phys. Mat. Klasse, 966 (1921).
- [2] O. Klein, Z. Phys. **37**, 895 (1926).
- [3] M. J. Duff and B.E.W. Nilsson, Phys. Rep. **130**, 1 (1986).
- [4] J.M. Overduin and P.S. Wesson, Phys. Rep. **283**, 305(1997).
- [5] X. Yang, Y. Ma, J. Shao, and W. Zhou, Phys. Rev. D**68**, 024006 (2003).
- [6] R.M. Wald, General relativity, (The University of Chicago Press, 1984).
- [7] J-P. Uzan, Rev. Mod. Phys. **75**, 403 (2003).
- [8] P.J.E. Peebles and B. Ratra, Rev. Mod. Phys. **75**, 599 (2003).
- [9] I.K. Wehus and F. Ravndal, "Dynamics of scalar field in 5-dimensional Kaluza-Klein theory". hep-ph/0210292.
- [10] N. Mohammadi, Phys. Rev. D**65**, 104018 (2002).
- [11] P.S. Wesson, Space-Time-Matter: Modern Kaluza-Klein Theory, (World Scientific Publishing, 1999).
- [12] M. Blagojevic, Gravitation and Gauge Symmetry, (IOP Publishing, 2002).
- [13] S.S. Seahra, Phys. Rev. D**65**, 124004 (2002).
- [14] J. Ponce de Leon, Grav. Cosmol. **8**, 272 (2002).
- [15] F. Dahia, E.M. Monte, and C. Romero, Mod. Phys. Lett. A**18**, 1773 (2003).
- [16] C. Liang, Introductory Differential Geometry and General Relativity I, II (Beijing Normal University Press, 2000, in Chinese).
- [17] S.W. Hawking and G.F.R. Ellis, The Large Scale Structure of Space-Time, (Cambridge University Press, 1980).