Kaluza-Klein Theory

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December 21, 2008

The purpose of this short write-up is simply to present the basic Kaluza-Klein theory, with particular attention given to the five-dimensional metric $\widetilde{g}_{\mu\nu}$ and the terms in the associated Ricci tensor $\widetilde{R}_{\mu\nu}$. There seems to be no explicit derivation of these terms on the Internet, so I thought it might help the interested student to sort things out. You may want to actually compute the various terms yourself and compare them with the ones presented here, if only to practice your facility with tensor-juggling (I did them by hand, then checked the results with MathTensor, so I believe they're correct).

You may also be interested to know that the original 1921 theory has evolved into today's string theory, as both share the idea of using multiple extra space dimensions to describe the world. The most advanced version of strings is known as M-theory, which utilizes an 11-dimensional spacetime having *seven* compactified extra space dimensions – a far cry from Kaluza's original *single* extra dimension!

By the way, Kaluza originally came up with his idea in 1919 and communicated it to Einstein in hope that the great scientist would recommend it for publication. But Einstein, who expressed great admiration for Kaluza's idea, sat on it for two years before recommending it. I'm sure this did not sit well with Kaluza.

Notation

The notational history of higher spacetimes is annoyingly confusing (like that of early tensor calculus), mainly because one normally denotes time as the "zeroeth" index and 1,2,3 for the space indices. Using "4" as the index for the fifth dimension seems to be problematic, but no consensus seems to have ever been reached by the scientific community. Here I will use "5" to denote the fifth dimension, so that time and the four space dimensions will go like 0,1,2,3,5, even though this notation then invites the question of what happened to the "4".

In denoting the usual 4-dimensional spacetime indices I will use Greek indices ($\mu, \nu = 0, 1, 2, 3$), while the full complement of five dimensions will be expressed by upper-case Latin letters ($A, B = 0, 1, 2, 3, 5$). When this is not sufficient, I will place a squiggle over the quantity to denote its five-dimensional provenance (as in $\widetilde{R}_{\mu\nu}$). This still does not provide total clarity (for example, the Ricci scalar in five dimensions is $\widetilde{R} = g^{AB} R_{AB}$, while its four-dimensional form $\widetilde{R} = \widetilde{g}^{\mu\nu}\widetilde{R}_{\mu\nu}$ looks the same).

I will denote ordinary partial differentiation with a single subscripted bar, as in

$$A_{\mu|\nu} = \partial_\nu A_\mu = \frac{\partial A_\mu}{\partial x^\nu}$$

while covariant differentiation will be denoted using a double bar, as in

$$F^{\lambda}_{\ \mu||\nu} = F^{\lambda}_{\ \mu|\nu} - F^{\lambda}_{\ \alpha} \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} + F^{\alpha}_{\ \mu} \left\{ \begin{matrix} \lambda \\ \alpha\nu \end{matrix} \right\}$$

where the terms in braces are the Christoffel symbols of the second kind.

Assumptions and Conventions

The primary assumption of the Kaluza-Klein theory (besides the fact that a fifth dimension actually exists) is the independence of all vector and tensor quantities with respect to the fifth coordinate. Consequently, we have identities such as $g_{AB|5} = 0, g_{\mu\nu|5} = 0, A_{\mu|5} = 0$, etc. This has come to be known as the "cylinder condition," since it implies that 4-dimensional spacetime lies along a cylindrical fifth dimension whose spacial extent is small enough to render it "invisible" to the underlying subspace. It was Klein who first postulated the idea that the fifth dimension is a cylindrical space having a radius roughly equal to the Planck length (10^{-35} meter), a concept that conveniently explains why the fifth dimension has not been directly observed. This same concept has been carried over to string theory, and it partially explains why this theory has been so difficult to verify. Indeed, a space having the dimensions of the Planck length would require energies equivalent to that of the Big Bang to resolve. If not overcome, this restriction may ultimately relegate string theory to a kind of unprovable religious faith.

The metric in five dimensions can be viewed symbolically as $g_{AB} = (g_{\mu\nu}, g_{\mu 5}, g_{55})$. The metric $g_{\mu\nu}$ represents the usual four-dimensional metric field, while the $g_{\mu 5}$ is a vector that is to be identified with the electromagnetic four-potential field A_μ. The remaining quantity g_{55} appears as a superfluous field; consequently, this field is usually normalized to unity, but I'm going to call it the constant k. Lastly, following both Kaluza and Klein, all the components of g_{AB} are considered constants under differentiation with respect to the fifth coordinate: $g_{AB|5} = 0$.

Introduction

Shortly after Einstein's November 1915 announcement of his theory of general relativity, physicists initiated efforts to generalize it in an attempt to develop a unified theory of the gravitational and electromagnetic forces (the only two forces known at the time). Notable among these efforts were those of Weyl (1918) and Kaluza (1921). In 1918, the German mathematical physicist Hermann Weyl used an intuitively-appealing version of non-Riemannian geometry to embed the entirety of electrodynamics into the affine connection of general relativity. Then in 1919 the Polish-German physicist Theodor Kaluza came up with another idea that employed ordinary Riemannian geometry but with five dimensions (one of time, and four of space). Einstein famously lauded Weyl's theory, but quickly withdrew his support when he discovered that the theory was not physical. Einstein similarly praised Kaluza's idea, although Einstein and other prominent physicists of the day were uncomfortable with the idea of a five-dimensional world (interestingly, Einstein himself played with five-dimensional relativity on and off for several decades, even after the scientific community had cooled toward the idea).

In 1926 the Swedish physicist Oskar Klein came up with some major improvements to Kaluza's theory, at which time it became universally known as Kaluza-Klein theory. But the theory languished for decades until the early advent of string theory in the 1970s, when serious interest in extra dimensions experienced a resurgence.

Kaluza's Basic Idea

In early 1919 Kaluza sent a letter to Einstein forwarding a draft paper that he hoped Einstein would endorse for publication. Kaluza's paper desc ribed how the inclusion of an additional space dimension in ordinary Riemannian geometry appeared to produce Maxwell's equations in free space. Kaluza's basic idea was as follows. Generalize the symmetric metric tensor $g_{\mu\nu}$ by adding an additional row and column with the quantities shown as follows:

$$g_{AB} = \begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} & kA_0 \\ g_{01} & g_{11} & g_{12} & g_{13} & kA_1 \\ g_{02} & g_{12} & g_{22} & g_{23} & kA_2 \\ g_{03} & g_{13} & g_{23} & g_{33} & kA_3 \\ kA_0 & kA_1 & kA_2 & kA_3 & k \end{bmatrix} \tag{1}$$

where A_μ is an as-yet undefined vector and k is a constant. If we now fully expand the five-dimensional form of the geodesics

$$\frac{d^2x^A}{ds^2} + \left\{ \begin{matrix} A \\ BD \end{matrix} \right\} \frac{dx^B}{ds}\frac{dx^D}{ds} = 0$$

(which can be derived by extremalizing the integral form of the line element $ds^2 = g_{AB}dx^A dx^B$), we get

$$\frac{d^2x^\lambda}{ds^2} + \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = -kF^\lambda_{\ \mu}\frac{dx^\mu}{ds}\frac{dx^5}{ds} - kg^{\lambda 5}A_{\mu|\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \tag{2}$$

where $F^\lambda_{\ \mu}$ is the upper-index form of the Maxwell tensor $F_{\mu\nu} = A_{\mu|\nu} - A_{\nu|\mu}$ (a similar expression results for the $A = 5$ geodesic). The first term on the right side looks just like the familiar Lorentz force term of a charged particle if we identify $k\, dx^5/ds$ with the charge to mass ratio e/m. Largely on the basis of this result, Kaluza believed the fifth dimension had something to do with electrodynamics, with the $g_{\mu 5}$ being identified with the electromagnetic four-potential and dx^5/ds as a kind of charge current.

Unfortunately, (2) contains an additional term involving $A_{\mu|\nu}$, which is not a tensor quantity. Consequently, Kaluza's g_{AB} cannot be a true tensor unless we set $g^{\lambda 5} = 0$, which is also problematic. Indeed, Kaluza did not identify the inverse metric field g^{AB} (and I have not been able to elucidate its form, either). Furthermore, Kaluza's form for the metric determinant $|g|$, which is necessary for the Kaluza action integral, is a mess.

Klein's Modification

Overall, Kaluza's basic idea seemed promising, but it exhibited problems, the biggest of which seems to be the fact that the metric is not a tensor. In 1926, the Swedish physicist Oskar Klein produced the first of two papers that seemed to alleviate this problem (remarkably, he seems to have been unaware of Kaluza's work). Klein asserted that the metric actually takes the forms

$$\widetilde{g}_{AB} = \begin{bmatrix} g_{00}+kA_0A_0 & g_{01}+kA_0A_1 & g_{02}+kA_0A_2 & g_{03}+kA_0A_3 & kA_0 \\ g_{01}+kA_0A_1 & g_{11}+kA_1A_1 & g_{12}+kA_1A_2 & g_{13}+kA_1A_3 & kA_1 \\ g_{02}+kA_0A_2 & g_{12}+kA_1A_2 & g_{22}+kA_2A_2 & g_{23}+kA_2A_3 & kA_2 \\ g_{03}+kA_0A_3 & g_{13}+kA_1A_3 & g_{23}+kA_2A_3 & g_{33}+kA_3A_3 & kA_3 \\ kA_0 & kA_1 & kA_2 & kA_3 & k \end{bmatrix} \tag{3}$$

$$\widetilde{g}^{AB} = \begin{bmatrix} g^{00} & g^{01} & g^{02} & g^{03} & -A^0 \\ g^{01} & g^{11} & g^{12} & g^{13} & -A^1 \\ g^{02} & g^{12} & g^{22} & g^{23} & -A^2 \\ g^{03} & g^{13} & g^{23} & g^{33} & -A^3 \\ -A^0 & -A^1 & -A^2 & -A^3 & 1/k + A_\mu A^\mu \end{bmatrix} \tag{4}$$

which, happily enough, give us the familiar identity $\widetilde{g}_{AB}\,\widetilde{g}^{AD} = \delta^D_B$. More importantly, the metric determinant of Klein's metric is

$$\widetilde{g} = kg \tag{5}$$

which can be verified by direct calculation. This, in itself, is something of a miracle: Klein's five-dimensional determinant $\widetilde{g}$ is independent of the vector field A_μ. *In my opinion, this is the key characteristic of the Klein metric.*

Klein's four-dimensional geodesic comes out as

$$\frac{d^2x^\lambda}{ds^2} + \begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = -kF^\lambda_{\ \mu}\frac{dx^\mu}{ds}\frac{dx^5}{ds} - \frac{1}{2}k\,A_\nu F^\lambda_{\ \mu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \tag{6}$$

which again reproduces the Lorentz force term. This expression is now fully covariant, although the $A_\nu F^\lambda_{\ \mu}$ term does not have any classical correspondence.

Identification of A_μ with Electrodynamics

So far we have no real reason to believe that the Kaluza-Klein vector A_μ is related to electromagnetism. In response to this, both Kaluza and Klein considered an infinitesimal change in the fifth coordinate,

$$\begin{aligned} x^5 &\rightarrow \overline{x}^5 = x^5 + \zeta(x^\mu), \qquad \text{or} \\ \delta dx^5 &= d\overline{x}^5 - dx^5 = \zeta_{|\mu} dx^\mu \end{aligned}$$

where ζ is some arbitrary scalar field such that $|\zeta| \ll 1$. The five-dimensional line element ds^2, given by

$$ds^2 = \widetilde{g}_{AB}dx^A dx^B = g_{\mu\nu}dx^\mu dx^\nu + 2kA_\mu dx^\mu dx^5 + kA_\mu A_\nu dx^\mu dx^\nu + k\left(dx^5\right)^2$$

must be invariant with respect to this variation. The subspace line element $g_{\mu\nu}dx^\mu dx^\nu$ is automatically invariant so we are left with

$$\delta ds^2 = 2kdx^\mu dx^5\,\delta A_\mu + 2kA_\mu dx^\mu\,\delta dx^5 + 2kA_\mu dx^\mu dx^\nu \delta A_\nu + 2kdx^5\,\delta dx^5$$

It is a simple matter to show that δds^2 vanishes if and only if the variation of the vector A_μ satisfies $\delta A_\mu = -\zeta_{|\mu}$; that is,

$$\overline{A}_\mu = A_\mu - \zeta_{|\mu} \tag{7}$$

This is the well-known *gauge transformation property* of the electromagnetic four-potential, and it strengthens the identification of the Kaluza-Klein vector A_μ with the electromagnetic field. This, together with the appearance of a Lorentz force-like term in the geodesic equations, provided both Kaluza and Klein a tempting, if still tentative, reason to believe that the fifth dimension has something to do with electrodynamics. But any hesitation these physicists may have had at this point was to be erased when Klein investigated the action Lagrangian for the theory.

The Kaluza-Klein Action

It is well-known that Einstein's path to the final expression of his theory of general relativity would have been significantly shortened if he had simply considered the action approach to gravitation, which lies in extremalizing the integral

$$I_G = \int \sqrt{-g}\, R\, d^4x \tag{8}$$

where $R = g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar. Variation of this integral with respect to the metric gives

$$\delta\, I_G = \int \sqrt{-g}\, \left[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right]\, \delta\, g^{\mu\nu}\, d^4x \tag{9}$$

from which we get the celebrated Einstein field equation for free space,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \tag{10}$$

Klein naturally a ssumed that the action would generalize in five dimensions via

$$\begin{aligned} I_{KK} &= \int \sqrt{-\widetilde{g}}\, \widetilde{R}\, d^5x && (11) \\ &= \sqrt{k} \int \sqrt{-g}\, \widetilde{R}\, d^5x && (12) \end{aligned}$$

But before Klein could actually perform the variation, he had to calculate all the terms in the five-dimensional Ricci scalar $\widetilde{R}$. This tedious but straightforward effort results in the following Christoffel identities, which I will just write down without bothering with the details:

$$\begin{aligned} \widetilde{\left\{ {\lambda \atop \mu\nu} \right\}} &= \left\{ {\lambda \atop \mu\nu} \right\} + \frac{1}{2} k \left[A_\mu F^\lambda_{\ \nu} + A_\nu F^\lambda_{\ \mu} \right], \qquad \widetilde{\left\{ {\lambda \atop \mu\lambda} \right\}} = \left\{ {\lambda \atop \mu\lambda} \right\} + \frac{1}{2} k\, A_\lambda F^\lambda_{\ \mu} \\ \widetilde{\left\{ {5 \atop \mu 5} \right\}} &= -\frac{1}{2} k\, A_\lambda F^\lambda_{\ \mu}, \qquad \widetilde{\left\{ {A \atop 55} \right\}} = \widetilde{\left\{ {\lambda \atop 55} \right\}} = 0, \qquad \widetilde{\left\{ {\lambda \atop \mu 5} \right\}} = \frac{1}{2} k\, F^\lambda_{\ \mu} \\ \widetilde{\left\{ {\lambda \atop \lambda 5} \right\}} &= 0, \qquad \widetilde{\left\{ {5 \atop \mu\nu} \right\}} = \frac{1}{2} \left[A_{\mu||\nu} + A_{\nu||\mu} \right] - \frac{1}{2} k\, A_\lambda \left[A_\mu\, F^\lambda_{\ \nu} + A_\nu\, F^\lambda_{\ \mu} \right] \end{aligned}$$

This done, we can now calculate the Ricci terms. The Ricci tensor in our five-dimensional notation is

$$\widetilde{R}_{AB} = \widetilde{\left\{ {D \atop AD} \right\}}_{|B} - \widetilde{\left\{ {D \atop AB} \right\}}_{|D} + \widetilde{\left\{ {D \atop AE} \right\}} \widetilde{\left\{ {E \atop BD} \right\}} - \widetilde{\left\{ {D \atop DE} \right\}} \widetilde{\left\{ {E \atop AB} \right\}}$$

from which we get (this is the only hard part!)

$$\begin{aligned} \widetilde{R}_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} k \left[A_\mu\, F^\lambda_{\ \nu||\lambda} + A_\nu\, F^\lambda_{\ \mu||\lambda} \right] + \frac{1}{4} k \left[F^\lambda_{\ \nu} F_{\lambda\mu} + F^\lambda_{\ \mu} F_{\lambda\nu} \right] - \frac{1}{4} k^2\, A_\mu A_\nu F_{\alpha\beta} F^{\alpha\beta}, \\ \widetilde{R}_{\mu 5} &= -\frac{1}{2} k\, F^\lambda_{\ \mu||\lambda} - \frac{1}{4} k^2\, A_\mu\, F_{\alpha\beta} F^{\alpha\beta}, \qquad \widetilde{R}_{55} = -\frac{1}{4} k^2\, F_{\alpha\beta} F^{\alpha\beta} \end{aligned}$$

Now, using

$$\widetilde{R} = \widetilde{g}^{AB}\, \widetilde{R}_{AB}$$

we get the simple result

$$\widetilde{R} = R + \frac{1}{4} k\, F_{\alpha\beta} F^{\alpha\beta}$$

Klein did this exact same calculation, and at this point he must have fallen off his chair. He immediately saw that, without even doing the variational calculation, the five-dimensional action (11) immediately leads to the correct four-dimensional action for the combined gravitational-electromagnetic field:

$$I_{KK} = \int \sqrt{-\widetilde{g}}\,\widetilde{R}\,d^5x \tag{13}$$

$$= \sqrt{k}\int \sqrt{-g}\,\left[R + \frac{1}{4}k\,F_{\alpha\beta}F^{\alpha\beta}\right]\,d^4x\,\int dx^5 \tag{14}$$

Klein was initially bothered by the integral term over dx^5(which gives infinity) but he quickly recognized that if the fifth dimension was cylindrical, x^5 could be viewed as an angular coordinate having the period $2\pi r$, where r is the cylinder's radius. Upon further consideration (which I won't repeat here), Klein determined that this radius must be on the order of the Planck constant. Klein thus concluded that the fifth dimension would be strictly unobservable.

The collapse of Klein's five-dimensional Lagrangian to four dimensions is an example of *dimensional reduction.* This phenomenon has proved to be a powerful tool in modern gauge theories, because a coordinate transformation in the higher space leads to a gauge transformation in the subspace. For the Kaluza-Klein model, the gauge transformation in (7) is the result of an infinitesimal transformation of the coordinate x^5.

Aftermath and Conclusions

The Kaluza-Klein model spurred considerable theoretical interest in the fifth dimension in the 1920s, and numerous physicists (including Einstein) tried to advance the theory, particularly with regard to the problem of matter and the possibility that gravity and the then-emerging field of quantum mechanics might somehow be connected via the fifth dimension. But in spite of its startling formal mathematical beauty, the theory made no new predictions with respect to gravity or electromagnetism, and the quantum connection seemed to lead nowhere. By the early 1930s, researchers had lost interest, and the Kaluza-Klein model had joined the ranks of other failed unified field theories.

In the early 1950s, Pauli tentatively proposed a six-dimensional Kaluza-Klein theory in an attempt to develop a non-abelian theory that would accommodate the weak and strong interactions. This too failed, and the concept of higher dimensions was pretty much scra pped until the 1970s, when string theory began to make its appearance. The first string theories described only bosons, and to accomplish this theorists had to assume the existence of 26 spacetime dimensions. Subsequent developments in string theory brought that number down to ten, but there were still problems involving uniqueness. In 1995, Witten showed that a consistent, unique theory of strings would require an additional spacial dimension, bringing the total to eleven. Since the elucidation of a single compactified spacial dimension would require energies far beyond what are now possible, the experimental detection of *seven* compactified spaces seems truly hopeless. Consequently, string theory may never be testable.

Nevertheless, the Kaluza-Klein approach shows that compactified extra dimensions lead naturally to locally gauge-invariant theories. If we view the fifth dimension x^5 as an angular coordinate Θ, then the smallness of the space makes the angle impossible to determine. The associated four-dimensional space sees this as a local symmetry, and indeed it is a type of *local gauge symmetry*, as Kaluza-Klein demonstrated for their assumed four-potential A_μ. This gauge-invariant aspect of compactified dimensions persists whether the extra dimensions are real (as in the Kaluza-Klein theory) or some kind of *internal degrees of freedom,* like particle spin.

The Kaluza-Klein theory is certainly tantalizing, and demonstrates that there may indeed be a profound connection between gravitation and electromagnetism, involving perhaps the fifth dimension. My personal feeling is that the theory's seemingly magical ability to produce the combined gravititational-electrodynamic Lagrangian may be nothing more than a lucky coincidence. The mathematical consistency of the Klein metric in five dimensions, however, tempts one into thinking that there may be something to extra dimensions after all. Hopefully, the renewed start-up of the European Large Hadron Collider, now scheduled for summer 2009, will resolve the issue once and for all.

References

1. P. Bergmann, *Introduction to the Theory of Relativity.* Dover Publications, 1942. Bergmann addresses the KK theory in detail, but as usual he speaks in *Bergmannese*, which means that he presents the subject using his own unintelligible mathematical notation. He does exactly the same thing for Einstein's gravity theory and Weyl's gauge invariant theory in this book, turning what should be straightforward stuff into an unreadable hodgepodge. Of historical interest only.

2. M. Blagojevic, *Gravitation and Gauge Symmetries.* Institute of Physics Publishing, 2002. Provides a great overview of the topic, along with a lengthy discussion of the Kaluza-Klein g_{55} term, treated as a variable scalar field.

3. T. Kaluza, *On the Unification Problem of Physics.* Sitz. Ber. Preuss. Akad. Wiss. (1921) 966. Reproduced in O'Raifertaigh's book.

4. O. Klein, *Quantum Theory and Five-Dimensional Relativity.* Zeit. f. Physik 37 (1926) 895. Reproduced in O'Raiferaigh's book.

5. L. O'Raifertaigh, *The Dawning of Gauge Theory.* Princeton Series in Physics, 1997. A wonderful book that includes English translations of the five-dimensional theories of Kaluza and Klein. Please note that my Eq. (1) corresponds to Eq. (2) in Kaluza's paper, while my Eq. (3) corresponds to Klein's Eqs. (3), (8) and (10).

6. W. Pauli, *Theory of Relativity.* Dover Publications, 1958. Pauli briefly discusses the KK theory in Supplementary Note 23.

7. W. Pauli, Correspondence to Abraham Pais, July and December 1953. In these informal letters to his friend and colleague Pais, Pauli explored the possibility of a six-dimensional Kaluza-Klein approach. The always-irrepressible Pauli begins the first letter with the note "Written down July 23-25 1953 in order to see how it looks." The letters are reproduced in O'Raifertaigh's book.

8. A. Zee, *Quantum Field Theory in a Nutshell.* Princeton University Press, 2003. See Chapter VIII.1 for a brief overview of the KK theory, along with some exercises.