1 Euclidean reflection groups

The goal of this chapter is to introduce Euclidean reflection groups. This will be done in two ways. First of all, examples of reflection groups, in the plane and in 3-space, are discussed in detail. Secondly, we provide, via a preliminary discussion of Weyl chambers and invariant theory, a suggestion of the beautiful structure theorems that hold for reflection groups, and that explain the interest in such groups. The dihedral and symmetric groups will receive particular attention.

1-1 Reflections and reflection groups

We shall work in ℓ-dimensional *Euclidean space* $\mathbb{E}$. In other words, $\mathbb{E} = \mathbb{R}^\ell$ where $\mathbb{R}^\ell$ has the usual inner product. More abstractly, $\mathbb{E}$ is a ℓ-dimensional vector space over $\mathbb{R}$ provided with a pairing

$$(\_,\_)\colon \mathbb{E} \times \mathbb{E} \to \mathbb{R}$$

satisfying:

(i) $(ax + by, z) = a(x, z) + b(y, z)$
(ii) $(x, y) = (y, x)$
(iii) $(x, x) \geq 0$ and $(x, x) = 0$ if and only if $x = 0$.

(In the above $x, y, z \in \mathbb{E}$, while $a, b \in \mathbb{R}$.)

We can define *reflections* either with respect to hyperplanes or vectors. First of all, given a hyperplane $H \subset \mathbb{E}$ through the origin, let $L =$ the line through the origin that is orthogonal to H. (So $\mathbb{E} = H \oplus L$.) Then, define the linear transformations $s_H\colon \mathbb{E} \to \mathbb{E}$

$$s_H \cdot x = x \quad \text{if } x \in H$$
$$s_H \cdot x = -x \quad \text{if } x \in L.$$

We can also define reflections with respect to vectors. This is the formulation we shall be using. Given $0 \neq \alpha \in \mathbb{E}$, let $H_\alpha \subset \mathbb{E}$ be the hyperplane

$$H_\alpha = \{x \mid (x, \alpha) = 0\}.$$

We then define the reflection $s_\alpha\colon \mathbb{E} \to \mathbb{E}$ by the rules

$$s_\alpha \cdot x = x \quad \text{if } x \in H_\alpha$$
$$s_\alpha \cdot \alpha = -\alpha.$$

Observe: Given $0 \neq k \in \mathbb{R}$, then $H_\alpha = H_{k\alpha}$ and $s_\alpha = s_{k\alpha}$.

We shall call H_α the *reflecting hyperplane* or *invariant hyperplane* of s_α. Here are some useful properties of s_α and H_α.

Properties:

(A-1) $s_\alpha \cdot x = x - \frac{2(x,\alpha)}{(\alpha,\alpha)}\alpha$ for all $x \in \mathbb{E}$
(A-2) s_α is *orthogonal*, i.e., $(s_\alpha \cdot x, s_\alpha \cdot y) = (x, y)$ for all $x, y \in \mathbb{E}$
(A-3) $\det s_\alpha = -1$
(A-4) If φ is an orthogonal automorphism of $\mathbb{E}$ then

$$\varphi \cdot H_\alpha = H_{\varphi\cdot\alpha}$$

$$\varphi s_\alpha \varphi^{-1} = s_{\varphi\cdot\alpha}.$$

To prove (A-1), check the effect of the RHS of the formula on H_α and on α. To prove (A-2), substitute formula (A-1) in $(s_\alpha \cdot x, s_\alpha \cdot y)$. To prove the first fact of (A-4), observe that $x \in H_\alpha$ implies $(\varphi \cdot x, \varphi \cdot \alpha) = (x, \alpha) = 0$. Hence, $\varphi \cdot H_\alpha \subset H_{\varphi\cdot\alpha}$. By comparing dimensions, we then have $\varphi \cdot H_\alpha = H_{\varphi\cdot\alpha}$. To prove the second fact of (A-4), check the effect of $\varphi s_\alpha \varphi^{-1}$ on $H_{\varphi\cdot\alpha} = \varphi \cdot H_\alpha$ and on $\varphi \cdot \alpha$.

Besides reflections, we also have the concept of a reflection group. Let

$$O(\mathbb{E}) = \{f : \mathbb{E} \to \mathbb{E} \text{ linear and } \big(f(\alpha), f(\beta)\big) = (\alpha, \beta) \text{ for all } \alpha, \beta\}$$

be the *orthogonal group* of $\mathbb{E}$. Given $W \subset O(\mathbb{E})$, we say that W is a Euclidean *reflection group* if W is generated, as a group, by its reflections. Two reflection groups $W \subset O(\mathbb{E})$ and $W' \subset O(\mathbb{E}')$ will be said to be *isomorphic* if there exists a linear isomorphism $f : \mathbb{E} \to \mathbb{E}'$ preserving inner products and conjugating W to W'. In other words,

$$\big(f(x), f(y)\big) = (x, y) \quad \text{for all } x, y \in \mathbb{E}$$

$$fWf^{-1} = W'.$$

A reflection group $W \subset O(\mathbb{E})$ is *reducible* if it can be decomposed as $W = W_1 \times W_2$, where both $W_1 \subset O(\mathbb{E})$ and $W_2 \subset O(\mathbb{E})$ are nontrivial subgroups generated by reflections from W. Otherwise a reflection group will be said to be *irreducible*. Our treatment of reflection groups will include a classification of those that are finite and irreducible.

The concept of a reflection as defined here can be generalized. Beginning in Chapter 14, we shall deal with *pseudo-reflections*, the extension of reflections to vector spaces over arbitrary fields. Our reason for beginning with Euclidean reflection groups is that they possess a theory all their own. In particular, we can use the trigonometry of the underlying Euclidean space to understand their structure. Section 1-4 provides a good illustration of this process.

Remark: We shall generally write our groups multiplicatively. The one exception to our multiplicative notation will be $\mathbb{Z}/n\mathbb{Z}$ for the cyclic group with n elements. Group actions on sets, $G \times S \to S$, are defined at the beginning of Appendix B. Such actions will be used extensively throughout this book. We use "$\cdot$" to denote group actions on a set. See properties (A-1), (A-2) and (A-4), above, for illustrations of this notation.